Abstract

This note contains technical details on material in Chapter 11 of Walsh (2010). It is shown a) that equation (11.11) is equivalent to the optimal money base rule (11.10), when in (11.11) the forecasts of the shocks are linear projections on the observed interest rate; b) how to derive equation (11.21), i.e., that the nominal interest rate securing that the actual money supply always is equal to the value securing the average inflation target (i.e., the nominal interest rate that uses the actual money supply as an intermediate target).

1 Equivalence of (11.11) and (11.10), when (11.11) involves linear projections of the shocks on the nominal interest rate

We have the resulting interest rate in the model with $b_t = \mu i_t$, and equations

\begin{align}
  y_t &= -\alpha i_t + u_t \quad (11.1) \\
  m_t &= -c i_t + y_t + v_t \quad (11.2) \\
  m_t &= b_t + hi_t + \omega_t \quad (11.7)
\end{align}
given as
\[ i_t = \frac{v_t - \omega_t + u_t}{\alpha + c + \mu + h}. \tag{11.9} \]

The resulting solution for output is given by
\[ y_t = -\alpha \frac{v_t - \omega_t + u_t}{\alpha + c + \mu + h} + u_t \]
\[ = \frac{(c + \mu + h) u_t - \alpha (v_t - \omega_t)}{\alpha + c + \mu + h}, \]
with the associated variance
\[ \sigma_y^2 = \frac{(c + \mu + h)^2 \sigma_u^2 + \alpha^2 \sigma_v^2 + \sigma_w^2)}{(\alpha + c + \mu + h)^2}. \]

Minimizing this with respect to \( \mu \) gives the first-order condition
\[ 2 (c + \mu + h) \sigma_u^2 (\alpha + c + \mu + h)^2 - 2 (\alpha + c + \mu + h) \left[ (c + \mu + h)^2 \sigma_u^2 + \alpha^2 \left( \sigma_v^2 + \sigma_w^2 \right) \right] = 0, \]
and thus
\[ 2 (c + \mu + h) \sigma_u^2 (\alpha + c + \mu + h)^2 - 2 (\alpha + c + \mu + h) \left[ (c + \mu + h)^2 \sigma_u^2 + \alpha^2 \left( \sigma_v^2 + \sigma_w^2 \right) \right] = 0, \]
\[ (c + \mu + h) \sigma_u^2 (\alpha + c + \mu + h) - (c + \mu + h)^2 \sigma_u^2 - \alpha^2 \left( \sigma_v^2 + \sigma_w^2 \right) = 0, \]
\[ (c + \mu + h) (\alpha + c + \mu + h) - (c + \mu + h)^2 - \alpha^2 \frac{(\sigma_v^2 + \sigma_w^2)}{\sigma_u^2} = 0, \]
\[ (c + \mu + h) [(\alpha + c + \mu + h) - (c + \mu + h)] - \alpha^2 \frac{(\sigma_v^2 + \sigma_w^2)}{\sigma_u^2} = 0, \]
and thus
\[ \mu^* = -(c + h) + \alpha \frac{(\sigma_v^2 + \sigma_w^2)}{\sigma_u^2}. \tag{11.10} \]

as the optimal coefficient in the policy rule \( b_t = \mu i_t \).

The alternative is that shocks are observable, so that the base rule can be stated as
\[ b_t = \mu_u u_t + \mu_v v_t + \mu_\omega \omega_t. \]

Inserting this into (11.1), (11.2) and (11.7) gives:
\[ m_t = \mu_u u_t + \mu_v v_t + \mu_\omega \omega_t + h i_t + \omega_t \]
\[ = -c i_t + y_t + v_t \]
Eliminating $i_t$ by (11.1) yields
\[ \mu_u u_t + \mu_v v_t + \mu_\omega \omega_t + h \frac{u_t - y_t}{\alpha} + \omega_t = -c \frac{u_t - y_t}{\alpha} + y_t + v_t \]
from which we get the solution for $y_t$:
\[ y_t \left( 1 + \frac{c + h}{\alpha} \right) = (\mu_v - 1) v_t + \left( \mu_u + \frac{c + h}{\alpha} \right) u_t + (1 + \mu_\omega) \omega_t \]
We immediately see that
\[ b_t = -\frac{c + h}{\alpha} u_t + v_t - \omega_t \]
completely stabilizes output. However, shocks cannot be observed, so estimates of the shocks are made based on the observed interest rate, so that the policy rule becomes
\[ b_t = -\frac{c + h}{\alpha} \hat{u}_t + \hat{v}_t - \hat{\omega}_t \]
where
\[ \hat{u}_t = \mathbb{E}[u_t|i_t], \quad \hat{v}_t = \mathbb{E}[v_t|i_t], \quad \hat{\omega}_t = \mathbb{E}[\omega_t|i_t]. \]
are the estimates of the shocks based on the observed interest rate. We assume that these estimates are made by linear projections on $i_t$ with the aim of minimizing the squared forecast errors.

But what is the nominal interest rate under this rule? From (11.1) and (11.2) one immediately gets
\[ m_t = -(c + \alpha) i_t + u_t + v_t \]
Using (11.7) one gets
\[ b_t + h \hat{i}_t + \omega_t = -(c + \alpha) i_t + u_t + v_t, \]
and thus
\[ i_t = -\frac{b_t}{\alpha + c + h} + \frac{u_t + v_t - \omega_t}{\alpha + c + h}. \]
As the shock forecasts are linear projections on $i$: \[ \hat{u}_t = \mathbb{E}[u_t|i_t] = \hat{\delta}_u i_t, \quad \hat{v}_t = \mathbb{E}[v_t|i_t] = \hat{\delta}_v i_t \] and \[ \hat{\omega}_t = \mathbb{E}[\omega_t|i_t] = \hat{\delta}_\omega i_t, \] where $\hat{\delta}_u$, $\hat{\delta}_v$ and $\hat{\delta}_\omega$ are the estimation coefficients to be determined, we have that
\[ b_t = \left( -\frac{c + h}{\alpha} \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega \right) i_t. \quad (11.11) \]
Therefore we get an expression for the interest rate from
\[ i_t (\alpha + c + h) = - \left( \frac{c + h \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega}{\alpha} \right) i_t + u_t + v_t - \omega_t, \]
so that the solution for the interest rate becomes
\[ i_t = \frac{u_t + v_t - \omega_t}{\alpha + c + h - \frac{c + h \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega}{\alpha}}. \]

Hence, the estimates are found from
\[ \hat{u}_t = \delta_u i_t \]
\[ = \frac{u_t + v_t - \omega_t}{\alpha + c + h - \frac{c + h \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega}{\alpha}}, \]
where \( \delta_u \) minimizes the squared forecast error. I.e., it solves
\[ \min_{\delta_u} E \left[ \hat{u}_t - u_t \right]^2 \]

The first-order condition is:
\[ 2\delta_u \frac{\sigma_v^2 + \sigma_\omega^2 + \sigma_u^2}{\left( \alpha + c + h - \frac{c + h \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega}{\alpha} \right)^2} - 2 \frac{\sigma_u^2}{\alpha + c + h - \frac{c + h \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega}{\alpha}} = 0, \]
or,
\[ \delta_u \frac{\sigma_v^2 + \sigma_\omega^2 + \sigma_u^2}{\alpha + c + h - \frac{c + h \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega}{\alpha}} - \sigma_u^2 = 0 \quad (\ast) \]

We then find the estimation coefficient \( \hat{\delta}_v \) from
\[ \min_{\delta_v} E \left[ \hat{v}_t - v_t \right]^2 \]
\[ = \min_{\delta_v} E \left[ \delta_v \frac{u_t + v_t - \omega_t}{\alpha + c + h - \frac{c + h \hat{\delta}_u + \hat{\delta}_v - \hat{\delta}_\omega}{\alpha}} - v_t \right]^2 \]

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The first-order condition becomes
\[ \delta_v \frac{\sigma^2_v + \sigma^2_\omega + \sigma^2_u}{\alpha + c + h - \frac{c+h}{\alpha} \delta_u + \delta_v - \delta_\omega} - \sigma^2_v = 0 \] (**) 
Likewise, the first order condition determining \( \delta_\omega \) becomes
\[ \delta_\omega \frac{\sigma^2_v + \sigma^2_\omega + \sigma^2_u}{\alpha + c + h - \frac{c+h}{\alpha} \delta_u + \delta_v - \delta_\omega} + \sigma^2_\omega = 0 \] (***) 
Combining (*) , (**) and (***) reveals that the estimates are
\[ \hat{\delta}_u = \alpha, \quad \hat{\delta}_v = \frac{\alpha \sigma^2_v}{\sigma^2_u}, \quad \hat{\delta}_\omega = -\frac{\alpha \sigma^2_\omega}{\sigma^2_u}. \]
Hence, the forecast-based rule for \( b \) is given by
\[ b_t = \left( -\frac{c + h}{\alpha} \delta_u + \delta_v - \delta_\omega \right) i_t \]
\[ = \left[ -(c + h) + \frac{\alpha (\sigma^2_v + \sigma^2_\omega)}{\sigma^2_u} \right] i_t \]
\[ = \mu^* i_t. \]
I.e., the exact same optimal rule as derived before by (11.10).

2 Deriving equation (11.21)

The model is given by
\[ y_t = a (\pi_t - E_{t-1} \pi_t) + z_t \]
\[ y_t = -\alpha (i_t - E_t \pi_{t+1}) + u_t \]
\[ m_t - p_t = m_t - \pi_t - p_{t-1} = y_t - ci_t + v_t \]
What money supply would give an inflation target of \( \pi^* \)? The trick is to acknowledge that with the “strict inflation-targeting preferences,” we have that inflation is on target on average. I.e., \( E_{t-1} \pi_t = E_t \pi_{t+1} = \pi^*. \]

\(^1\)So, in contrast with the lecture slides, I here operate with an inflation target \( \pi^* \neq 0 \) to make the equations identical to Walsh’s.
the model is rewritten as

\[\begin{align*}
y_t &= a (\pi_t - \pi^*) + z_t \\
y_t &= -\alpha (i_t - \pi^*) + u_t \\
m_t - p_t &= m_t - \pi_t - p_{t-1} = y_t - ci_t + v_t
\end{align*}\]

Now, the LM curve is inserted into the IS curve to eliminate \(i_t\):

\[\begin{align*}
y_t &= -\alpha \frac{y_t + v_t - m_t + \pi_t + p_{t-1}}{c} + \alpha \pi^* + u_t,
\end{align*}\]

and

\[\begin{align*}
y_t \left(1 + \frac{\alpha}{c}\right) &= \alpha \frac{-v_t + m_t - \pi_t - p_{t-1}}{c} + \alpha \pi^* + u_t, \\
y_t \frac{\alpha + c}{c} &= \alpha \frac{-v_t + m_t - \pi_t - p_{t-1}}{c} + \alpha \pi^* + u_t, \\
y_t &= \frac{\alpha}{\alpha + c} \left(m_t - \pi_t - p_{t-1} - v_t\right) + \frac{c}{\alpha + c} \left(\alpha \pi^* + u_t\right).
\end{align*}\]

We then find the actual inflation rate by combining this expression with the “modified” Lucas supply schedule:

\[\begin{align*}
\frac{\alpha}{\alpha + c} \left(m_t - \pi_t - p_{t-1} - v_t\right) + \frac{c}{\alpha + c} \left(\alpha \pi^* + u_t\right) &= a (\pi_t - \pi^*) + z_t,
\end{align*}\]

from which we get the solution for the inflation rate for a given money supply:

\[\begin{align*}
\pi_t \left(a + \frac{\alpha}{\alpha + c}\right) &= \frac{\alpha}{\alpha + c} \left(m_t - p_{t-1} - v_t\right) + \frac{c}{\alpha + c} \left(\alpha \pi^* + u_t\right) + a \pi^* - z_t \\
\pi_t \frac{a(\alpha + c) + \alpha}{\alpha + c} &= \frac{\alpha}{\alpha + c} \left(m_t - p_{t-1} - v_t\right) + \frac{c}{\alpha + c} \left(\alpha \pi^* + u_t\right) + a \pi^* - z_t
\end{align*}\]

and therefore

\[\begin{align*}
\pi_t &= \frac{\alpha}{a(\alpha + c) + \alpha} \left(m_t - p_{t-1} - v_t\right) + \frac{c}{a(\alpha + c) + \alpha} \left(\alpha \pi^* + u_t\right) + \frac{a(\alpha + c) \pi^* - (\alpha + c) z_t}{a(\alpha + c) + \alpha},
\end{align*}\]

We then solve for the value of \(m_t\) that secures \(\pi_t = \pi^*\). I.e., this value must satisfy

\[\begin{align*}
\pi^* &= \frac{\alpha}{a(\alpha + c) + \alpha} \left(m_t - p_{t-1} - v_t\right) + \frac{c}{a(\alpha + c) + \alpha} \left(\alpha \pi^* + u_t\right) + \frac{a(\alpha + c) \pi^* - (\alpha + c) z_t}{a(\alpha + c) + \alpha},
\end{align*}\]
from which we get

\[
\pi^* \left[ 1 - \frac{c \alpha + a (\alpha + c)}{a (\alpha + c) + \alpha} \right] = \frac{\alpha}{a (\alpha + c) + \alpha} \left( m_t - p_{t-1} - v_t \right) + \frac{c}{a (\alpha + c) + \alpha} u_t
\]

and

\[
\pi^* \frac{a (\alpha + c) + \alpha - ca - a (\alpha + c)}{a (\alpha + c) + \alpha} = \frac{\alpha}{a (\alpha + c) + \alpha} \left( m_t - p_{t-1} - v_t \right) + \frac{c}{a (\alpha + c) + \alpha} u_t - \frac{(\alpha + c)}{a (\alpha + c) + \alpha} z_t,
\]

\[
\pi^* \frac{\alpha (1 - c)}{a (\alpha + c) + \alpha} = \frac{\alpha}{a (\alpha + c) + \alpha} \left( m_t - p_{t-1} - v_t \right) + \frac{c}{a (\alpha + c) + \alpha} u_t - \frac{(\alpha + c)}{a (\alpha + c) + \alpha} z_t,
\]

and finally

\[
m_t = p_{t-1} + v_t + (1 - c) \pi^* - \frac{c}{\alpha} u_t + \frac{\alpha + c}{\alpha} z_t.
\]

As shocks are unobservable, the optimal target of \( m_t \) is given by

\[
\hat{m}_t = p_{t-1} + v_t + (1 - c) \pi^* - \frac{c}{\alpha} u_t + \frac{\alpha + c}{\alpha} z_{t-1}
\]

(11.19)

The actual money supply, for a given interest rate \( \hat{\nu}_t \), follows from the LM curve as

\[
m_t \left( \hat{\nu}_t \right) = \pi_t \left( \hat{\nu}_t \right) + p_{t-1} + y_t \left( \hat{\nu}_t \right) - c \hat{\nu}_t + v_t.
\]

(\*)

Note that we have that

\[
\hat{\nu}_t = \pi^* + \frac{1}{\alpha} (\rho_u u_{t-1} - \rho_z z_{t-1})
\]

(11.17)

and

\[
\pi_t \left( \hat{\nu}_t \right) = \pi^* + \frac{\varphi_t - e_t}{a}
\]

(11.18)

We can then find \( y_t \left( \hat{\nu}_t \right) \) by inserting \( \pi_t \left( \hat{\nu}_t \right) \) into the Lucas supply schedule:

\[
y_t \left( \hat{\nu}_t \right) = a \left( \pi^* + \frac{\varphi_t - e_t}{a} - \pi^* \right) + z_t
\]

\[
= \varphi_t - e_t + z_t
\]

\[
= \varphi_t + \rho_z z_{t-1}
\]
Then insert the found expressions for $\pi_t(\hat{i}_t)$, $y_t(\hat{i}_t)$ and $\hat{i}_t$ into (*):

$$
m_t \left( \hat{i}_t \right) = \pi^* + \frac{\varphi_t - \varepsilon_t}{a} + p_{t-1} + \varphi_t + \varphi z_{t-1} - c \left[ \pi^* + \frac{1}{\alpha} (\varphi u_{t-1} - \varphi z_{t-1}) \right] + \psi_t
$$

$$
= (1 - c) \pi^* + p_{t-1} + \psi_t + \frac{1 + a}{a} \varphi_t - \frac{c}{\alpha} \varphi u_{t-1} - \frac{1}{a} \varepsilon_t + \frac{c + a}{\alpha} \varphi z_{t-1}
$$

Note that

$$\hat{m}_t = p_{t-1} + (1 - c) \pi^* - \frac{c}{\alpha} \varphi u_{t-1} + \frac{\alpha + c}{\alpha} \varphi u_{t-1} + \varphi v_{t-1} \tag{11.19}$$

applied to (**) yields

$$
m_t \left( \hat{i}_t \right) = \hat{m}_t - \varphi v_{t-1} + \psi_t + \frac{1 + a}{a} \varphi_t - \frac{1}{a} \varepsilon_t \tag{11.20}
$$

Now, when actual $m_t$, conditional on $\hat{i}_t$, deviates from $\hat{m}_t$, it is time to change $i_t$ such that $m_t = \hat{m}_t$ again. What value of the interest rate will accomplish that? I.e., how do we derive equation (11.21) on page 525 in Walsh (2010)?

The trick is to solve the model for $m_t$ as a function of any value of the interest rate, and then find the interest rate that delivers $m_t = \hat{m}_t$. This can be accomplished by the central bank, as it observes $m_t$ even though it doesn’t observe the various period-$t$ disturbances.

As the model is

$$
y_t = a (\pi_t - \pi^*) + z_t
$$
$$
y_t = -\alpha (i_t - \pi^*) + u_t
$$
$$
m_t - p_t = m_t - \pi_t - p_{t-1} = y_t - \alpha i_t + u_t
$$

we first combine the AS and IS curve to find inflation as a function of the interest rate:

$$a (\pi_t - \pi^*) + z_t = -\alpha (i_t - \pi^*) + u_t$$

and thus

$$\pi_t = \frac{a + \alpha}{a} \pi^* - \frac{\alpha}{a} i_t + \frac{1}{a} (u_t - z_t)$$

We have output a function of the interest rate directly from the IS curve:

$$y_t = -\alpha (i_t - \pi^*) + u_t$$
We can use this in the LM relationship to find

\[
m_t = \frac{a + \alpha}{a} \pi^* - \frac{a + \alpha}{a} i_t + \frac{1}{a} \left( u_t - z_t \right) + p_{t-1} - \alpha (i_t - \pi^*) + u_t - c_i + v_t
\]

\[
= -\alpha \left[ (1 + a) + ca \right] i_t + \frac{a + \alpha + \alpha a}{a} \pi^* + p_{t-1} + \frac{1 + a}{a} u_t + v_t - \frac{1}{a} z_t
\]

Securing that \( m_t = \hat{m}_t \) requires that we use (11.19) and find the value of \( i_t \) that secures this equality. I.e.,

\[
\frac{-\alpha (1 + a) + ca}{a} i_t + \frac{a + \alpha + \alpha a}{a} \pi^* + p_{t-1} + \frac{1 + a}{a} u_t + v_t - \frac{1}{a} z_t
\]

\[
= p_{t-1} + \rho_v v_{t-1} + (1 - c) \pi^* - \frac{c}{a} \rho_u u_{t-1} + \frac{\alpha + c}{\alpha} \rho_z z_{t-1},
\]

must hold, or,

\[
\frac{-\alpha (1 + a) + ca}{a} i_t + \frac{a + \alpha + \alpha a}{a} \pi^* + p_{t-1} + \frac{1 + a}{a} u_t + v_t - \frac{1}{a} z_t
\]

\[
= \rho_v v_{t-1} + (1 - c) \pi^* - \frac{c}{a} \rho_u u_{t-1} + \frac{\alpha + c}{\alpha} \rho_z z_{t-1},
\]

\[
-\frac{\alpha (1 + a) + ca}{a} i_t + \frac{a + \alpha + \alpha a}{a} \pi^* + p_{t-1} + \frac{1 + a}{a} u_t + v_t - \frac{1}{a} z_t
\]

\[
+ \frac{1 + a}{a} \varphi_t + \psi_t - \frac{1}{a} \rho_z z_{t-1} - \frac{1}{a} e_t
\]

\[
= (1 - c) \pi^* - \frac{c}{a} \rho_u u_{t-1} + \frac{\alpha + c}{\alpha} \rho_z z_{t-1},
\]

\[
-\frac{\alpha (1 + a) + ca}{a} i_t + \frac{a + \alpha + \alpha a}{a} \pi^* + \left[ \frac{1 + a}{a} + \frac{c}{\alpha} \right] \rho_u u_{t-1}
\]

\[
+ \frac{1 + a}{a} \varphi_t + \psi_t - \left( \frac{1}{a} + \frac{\alpha + c}{\alpha} \right) \rho_z z_{t-1} - \frac{1}{a} e_t
\]

\[
= (1 - c) \pi^*
\]

\[
\left( \frac{a + \alpha + \alpha a}{a} + c - 1 \right) \pi^* + \left[ \frac{1 + a}{a} + \frac{c}{\alpha} \right] \rho_u u_{t-1}
\]

\[
+ \frac{1 + a}{a} \varphi_t + \psi_t - \left( \frac{1}{a} + \frac{\alpha + c}{\alpha} \right) \rho_z z_{t-1} - \frac{1}{a} e_t
\]

\[
= \frac{\alpha (1 + a) + ca}{a} i_t,
\]
\[
\left( \frac{a + \alpha + \alpha a}{a} + c - 1 \right) \pi^* + \frac{\alpha (1 + a) + ca}{a \alpha} \rho_u u_{t-1}
\]
\[+ \frac{1 + a}{a} \varphi_t + \psi_t - \frac{\alpha + a (\alpha + c)}{a \alpha} \rho_z z_{t-1} - \frac{1}{a} \epsilon_t
\]
\[= \frac{\alpha (1 + a) + ca}{a} i_t.
\]

Therefore,
\[
i_t \frac{\alpha (1 + a) + ca}{a} = \frac{\alpha (1 + a) + ca}{a} \pi^*
\]
\[+ \frac{\alpha (1 + a) + ca}{a \alpha} (\rho_u u_{t-1} - \rho_z z_{t-1})
\]
\[+ \frac{1 + a}{a} \varphi_t + \psi_t - \frac{1}{a} \epsilon_t,
\]

which finally gives
\[
i_t = \pi^* + \frac{1}{\alpha} (\rho_u u_{t-1} - \rho_z z_{t-1})
\]
\[+ \frac{(1 + a) \varphi_t - \epsilon_t + a \psi_t}{\alpha (1 + a) + ca}.
\]

Using the result for \( \hat{i}_t \), equation (11.17), this readily reduces to
\[
i_t = \hat{i}_t + \frac{(1 + a) \varphi_t - \epsilon_t + a \psi_t}{\alpha (1 + a) + ca} \equiv i_t^T
\]
which is equation (11.21) in Walsh (2010).