Deriving π_t on p. 31 in Galí (2008): Application of the Method of Undetermined Coefficients

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1 The equations to start with

At the exercises on November 29, we derived two expressions in the classical model with money in the utility function. One for output as a function of the nominal interest rate, and another for asset market equilibrium, "amended" so as to take into account that real money balances may affect the marginal utility of consumption. These equations were, respectively,

$$\begin{aligned} y_t &= \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}a_t - \frac{\omega(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha}i_t, \\ &\equiv \psi_{ya}a_t - \psi_{yi}i_t, \qquad (1) \\ y_t &= \mathrm{E}_t\{y_{t+i}\} - \sigma^{-1}\left[i_t - \mathrm{E}_t\{\pi_{t+1}\} - \rho - \omega\mathrm{E}_t\{\Delta i_{t+1}\}\right] \qquad (2) \end{aligned}$$

where

$$\omega \equiv \frac{k_m\beta\left(1-\dfrac{\sigma}{\nu}\right)}{1+k_m(1-\beta)}, \qquad k_m \equiv \frac{M/P}{C},$$

and where it follows that $\nu > \sigma$ implies $\omega > 0$. It is assumed that the nominal interest rate is determined according to the rule

$$i_t = \rho + \phi_\pi \pi_t + v_t, \qquad \phi_\pi > 1. \tag{3}$$

Furthermore, the exogenous disturbances follow AR(1) processes:

$$\begin{aligned} a_t &= \rho_a a_{t-1} + \varepsilon_t^a, \\ v_t &= \rho_v v_{t-1} + \varepsilon_t^v. \end{aligned}$$

2 Finding the relevant difference equation

The solution proceeds by inserting (3) into (1) and (2):

$$\begin{aligned} y_t &= \psi_{ya} a_t - \psi_{yi} \left(\rho + \phi_\pi \pi_t + v_t\right), \\ y_t &= \mathrm{E}_t \left\{y_{t+i}\right\} - \sigma^{-1} \left[\phi_\pi \pi_t + v_t - \mathrm{E}_t \left\{\pi_{t+1}\right\} - \omega \mathrm{E}_t \left\{\phi_\pi \Delta\pi_{t+1} + \Delta v_{t+1}\right\}\right]. \end{aligned}$$

Using the former in the latter yields

$$\begin{aligned} & \psi_{ya} a_t - \psi_{yi} \left(\rho + \phi_\pi \pi_t + v_t\right) \\ = \; & \mathrm{E}_t \left\{\psi_{ya} a_{t+1} - \psi_{yi} \left(\rho + \phi_\pi \pi_{t+1} + v_{t+1}\right)\right\} \\ & -\sigma^{-1} \left[\phi_\pi \pi_t + v_t - \mathrm{E}_t \left\{\pi_{t+1}\right\} - \omega \mathrm{E}_t \left\{\phi_\pi \Delta\pi_{t+1} + \Delta v_{t+1}\right\}\right]. \end{aligned}$$

Solving out expectations terms using the properties of the shock processes:

$$\begin{aligned} & \psi_{ya} a_t - \psi_{yi} \left(\rho + \phi_\pi \pi_t + v_t\right) \\ = \; & \psi_{ya} \rho_a a_t - \psi_{yi} \left(\rho + \phi_\pi \mathrm{E}_t \left\{\pi_{t+1}\right\} + \rho_v v_t\right) \\ & -\sigma^{-1} \left[\phi_\pi \pi_t + v_t - \mathrm{E}_t \left\{\pi_{t+1}\right\} - \omega \phi_\pi \mathrm{E}_t \left\{\Delta\pi_{t+1}\right\} - \omega \phi_\pi \left(\rho_v - 1\right) v_t\right]. \end{aligned}$$

Rearranging:

$$\begin{aligned} & -\sigma \psi_{ya} a_t + \sigma \psi_{yi} \left(\rho + \phi_\pi \pi_t + v_t\right) \\ = \; & -\sigma \psi_{ya} \rho_a a_t + \sigma \psi_{yi} \left(\rho + \phi_\pi \mathrm{E}_t \left\{\pi_{t+1}\right\} + \rho_v v_t\right) \\ & +\phi_\pi \pi_t + v_t - \mathrm{E}_t \left\{\pi_{t+1}\right\} - \omega \phi_\pi \mathrm{E}_t \left\{\pi_{t+1}\right\} + \omega \phi_\pi \pi_t + \omega \phi_\pi \left(1 - \rho_v\right) v_t. \end{aligned}$$

Collecting terms:

$$\begin{aligned} & -\sigma \psi_{ya} \left(1 - \rho_a\right) a_t - \left(1 + \left(1 - \rho_v\right) \left(\omega - \sigma \psi_{yi}\right)\right) v_t \\ = \; & \phi_\pi \left(1 + \omega - \sigma \psi_{yi}\right) \pi_t - \left(1 + \phi_\pi \left(\omega - \sigma \psi_{yi}\right)\right) \mathrm{E}_t \left\{\pi_{t+1}\right\}. \end{aligned}$$

Note that

$$\begin{aligned}
\omega - \sigma\psi_{yi} &= \omega - \sigma\frac{\omega\,(1-\alpha)}{\sigma\,(1-\alpha)+\varphi+\alpha}, \\
&= \omega\left(1 - \frac{\sigma\,(1-\alpha)}{\sigma\,(1-\alpha)+\varphi+\alpha}\right), \\
&= \omega\frac{\varphi+\alpha}{\sigma\,(1-\alpha)+\varphi+\alpha}, \\
&\equiv \omega\psi.
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
& -\sigma\psi_{ya}\,(1-\rho_a)\,a_t - (1+(1-\rho_v)\,\omega\psi)\,v_t \\
= \;& \phi_\pi\,(1+\omega\psi)\,\pi_t - (1+\phi_\pi\omega\psi)\,\mathrm{E}_t\,\{\pi_{t+1}\}\,,
\end{aligned}$$

leading to a first-order rational expectations difference equation in π_t:

$$\begin{aligned}
\pi_t &= \frac{1+\phi_\pi\omega\psi}{\phi_\pi\,(1+\omega\psi)}\mathrm{E}_t\,\{\pi_{t+1}\} - \frac{\sigma\psi_{ya}\,(1-\rho_a)}{\phi_\pi\,(1+\omega\psi)}a_t - \frac{1+(1-\rho_v)\,\omega\psi}{\phi_\pi\,(1+\omega\psi)}v_t, \\
\pi_t &= \Theta\mathrm{E}_t\,\{\pi_{t+1}\} - \frac{\sigma\psi_{ya}\,(1-\rho_a)}{\phi_\pi\,(1+\omega\psi)}a_t - \frac{1+(1-\rho_v)\,\omega\psi}{\phi_\pi\,(1+\omega\psi)}v, \qquad (4)
\end{aligned}$$

where

$$\Theta \equiv \frac{1+\phi_\pi\omega\psi}{\phi_\pi\,(1+\omega\psi)}.$$

This has a unique stationary solution if

$$-1 < \Theta < 1 \qquad (5)$$

In the case where $\omega > 0$, this is clearly satisfied as $\phi_\pi > 1$. However, in the case where $\omega < 0$ we cannot be sure that it holds (for $\omega\psi \to -1$, $\Theta \to -\infty$). We assume that (5) holds in the following.

3 Solving for π_t

We can now solve (4) for π_t by forward substitution. This is cumbersome, however, so instead we will solve (4) by the *method of undetermined coefficients*. This method involves two simple steps. In the first, one makes a conjecture about the form of the solution as a function of unknown coefficients. In the second step, one uses the conjecture together with the difference equation to verify the validity of the conjecture and to identify the unknown coefficients (which then implies that a solution is obtained).

In this case (and in all related cases of linear rational expectations models), it is natural to conjecture that inflation is a linear function of the shocks a_t and v_t. I.e.,

$$\pi_t = -Aa_t - Bv_t \tag{6}$$

where A and B are the undetermined coefficients to be identified. Forward (6) one period and take expectations:

$$\begin{aligned} \mathrm{E}_t\left\{\pi_{t+1}\right\} &= -A\mathrm{E}_t\left\{a_{t+1}\right\} - B\mathrm{E}_t\left\{v_{t+1}\right\} \\ \mathrm{E}_t\left\{\pi_{t+1}\right\} &= -A\rho_a a_t - B\rho_v v_t \end{aligned} \tag{7}$$

We now combine (7) with (4) to see whether the conjectured form of the solution is correct:

$$\begin{aligned} \pi_t &= \Theta\left[-A\rho_a a_t - B\rho_v v_t\right] \\ &\quad -\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)}a_t - \frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)}v. \end{aligned} \tag{8}$$

Wee see that it is. The conjecture is consistent with the difference equation; i.e., inflation *is* a linear function of the shocks. We can then identify A and B by using (6) together with (8):

$$\begin{aligned} -Aa_t - Bv_t &= \Theta\left[-A\rho_a a_t - B\rho_v v_t\right] \\ &\quad -\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)}a_t - \frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)}v \end{aligned}$$

and note that this must hold for *all* values of a_t and v_t. Hence, the following equations apply:

$$-A = -\Theta A\rho_a - \frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)}, \tag{9}$$

$$-B = -\Theta B\rho_v - \frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)}. \tag{10}$$

From (9) and (10) we get

$$\begin{aligned} A &= \frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}, \\ B &= \frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}, \end{aligned}$$

respectively, giving us the solution for inflation as

$$\pi_t = -\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t - \frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t, \tag{11}$$

which is precisely the one shown in Galí (2008), p. 31. The solution for i_t follows immediately by inserting (11) into (3) (Galí ignores the constant ρ on p. 31). Finally the solution for y_t follows by inserting the solution for i_t into (1).