Complete financial markets and consumption risk sharing

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This note shows in detail how an assumption of complete financial markets (in a sense to be defined) can imply consumption risk-sharing in the New-Keynesian model of Galí (2008, Chapters 6 and 7). The note only explains notation if it deviates from that of Galí, or if it is of particular importance for the subject at hand.

1 Complete financial markets

Complete financial markets means that one can purchase securities whose payments are conditional on the future state of the nature in the economy, and where there exist a security relating to any possible state of nature. Securities with these properties are often denoted Arrow-Debreu securities. They have the properties that they are purchased in period t at the price $V(s_{t+1}|s_t)$ where $s_t,\ s_{t+1} \in S$ are states of nature in period t and $t+1$, respectively, and S is the set of all possible states. The securities pay out zero in period $t+1$ if "their" particular state is not realized, and one if it is. The probability of a particular state happening in period $t+1$, given the state in t, is denoted $\xi_{t,t+1}^{s_{t+1}}$. An Arrow security purchased in period t pertaining to this state is denoted by the random variable $D\left(s_{t+1}|s_t\right)$.

With these assumptions, the market value of a portfolio of securities acquired in period t is given by

$$\sum_{s_{t+1}\in S} V\left(s_{t+1}|s_t\right) D\left(s_{t+1}|s_t\right).$$

We can write this in terms of an expected value, by noting that it equals

$$\sum_{s_{t+1}\in S} \xi_{t,t+1}^{s_{t+1}} \frac{V\left(s_{t+1}|s_t\right)}{\xi_{t,t+1}^{s_{t+1}}} D\left(s_{t+1}|s_t\right),$$

which can be written as

$$\mathrm{E}_t\left\{\frac{V\left(s_{t+1}|s_t\right)}{\xi_{t,t+1}^{s_{t+1}}} D\left(s_{t+1}|s_t\right)\right\}$$

where the expectation is taken over all states. For any state, define

$$Q_{t,t+1} \equiv \frac{V\left(s_{t+1}|s_t\right)}{\xi_{t,t+1}}.$$

This variable is normally called the *stochastic discount factor* as it is the relevant discount factor to value a stochastic, state-contingent security. The expected value of a stochastic portfolio in period t is then written as

$$\mathrm{E}_t\left\{Q_{t,t+1} D_{t+1}\right\} \quad \left(= \sum_{s_{t+1}\in S} \xi_{t,t+1}^{s_{t+1}} Q_{t,t+1} D\left(s_{t+1}|s_t\right)\right)$$

where the expectations are taken over all possible future states (and the notation is accordingly simplified in Galí, 2008).

2 Optimal goods demand and savings behavior by households under complete markets

Assume the following utility function of household j:

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t\left(j\right), N_t\left(j\right)\right).$$

With the complete markets assumptions, the budget constraint of consumer j is

$$P_t C_t + \mathrm{E}_t \left\{ Q_{t,t+1} D_{t+1}(j) \right\} \leq D_t(j) + W_t(j) L_t(j) - T_t, \qquad t \geq 0, \tag{1}$$

The relevant optimality conditions are

$$U_{c,t} = \lambda_t(j) P_t, \tag{2}$$
$$U_{n,t} = -\lambda_t(j) W_t(j), \tag{3}$$
$$\lambda_t(j) Q_{t,t+1} = \beta \lambda_{t+1}(j), \tag{4}$$

where $\lambda_t(j)$ is the Lagrange multiplier on (1), and where (4) holds for each possible future state of nature. I.e., $\lambda_{t+1}(j)$ should be understood as the multiplier in period $t+1$ in a particular state s_{t+1}.[1] As usual, this multiplier is interpreted as the marginal utility of nominal income. The assumption of complete markets is crucial for this multiplier across individuals in the economy. As all agents operate on the same asset market, they face the same stochastic discount factor. Hence, the growth rate in the marginal utility of income will be equalized across individuals in any state. I.e.,

$$\frac{\lambda_{t+1}(i)}{\lambda_t(i)} = \frac{\lambda_{t+1}(j)}{\lambda_t(j)} = \beta^{-1} Q_{t,t+1}, \tag{5}$$

for any pair of i,j in the population. This immediately shows the insurance effect of having access to complete markets: Individuals trade in state-contingent securities such that any differences in marginal utilities of income accross periods and states are eliminated.

Now, to facilitate aggregation, which here means that one can operate with a representative consumer in the model, one usually assumes symmetric initial conditions, i.e.,

$$\lambda_0(i) = \lambda_0(j) \tag{6}$$

for any pair of i,j in the population. Combining (6) and (5) one then gets the central risk-sharing result that the marginal utility nominal income are equalized across individuals in any period:

$$\lambda_t(j) \equiv \lambda_t \quad \text{all } j, \quad t \geq 1. \tag{7}$$

[1] Note that the probabilities do not appear, as they cancel out on the left-hand side (wthere they weigh the cost of the particular security), and the right-hand side where they appear multiplicatively to capture the probability of getting the return "one" in that particular state.

3 Implications for individual consumption under particular assumptions

When we apply the standard specification of utility used in Galí (2008), this result has implications for consumption across individuals. Let

$$U\left(C_t\left(j\right), N_t\left(j\right)\right) = \frac{1}{1-\sigma}\left(C_t\left(j\right)\right)^{1-\sigma} - \frac{1}{1+\varphi}\left(N_t\left(j\right)\right)^{1+\varphi}.$$

Under this specification, the marginal utility of consumption is independent of employment, so for agent j we have from (2) that

$$\left(C_t\left(j\right)\right)^{-\sigma} = \lambda_t\left(j\right) P_t$$

which by use of (7) becomes

$$\left(C_t\left(j\right)\right)^{-\sigma} = \lambda_t P_t$$

implying

$$C_t\left(j\right) \equiv C_t \quad \text{all } j.$$

Hence, complete asset markets and separability of the utility function between consumption and employment secure that all individuals have the same consumption levels in all periods; potential differences are eliminated by the insurance properties of asset markets. This is a result known as *full consumption risk sharing*, and is used by Galí (2008, p. 124) to ease derivation of the optimal wage in the case of the sticky-wage model.

As for the optimal allocation of consumption over time the first-order conditions can be combined into the conventional consumption-Euler equation, with the aid of yet another definition. Let $Q_t \equiv \mathrm{E}_t\left\{Q_{t,t+1}\right\}$ be the price of a one-period risk-less discount bond paying off one in period $t+1$ (hence, $i_t \equiv -\log Q_t$ is the nominal yield). Then, by taking expectations over (4) and using (7), we get

$$\lambda_t \mathrm{E}_t\left\{Q_{t,t+1}\right\} = \beta \mathrm{E}_t\left\{\lambda_{t+1}\right\},$$

or,

$$\lambda_t Q_t = \beta \mathrm{E}_t\left\{\lambda_{t+1}\right\}.$$

(This first-order condition would have emerged immediately if we had included this risk-less asset in the budget constraint.) Using that $C_t^{-\sigma} = \lambda_t P_t$ we then get

$$\frac{C_t^{-\sigma}}{P_t} Q_t = \beta \mathrm{E}_t\left\{\frac{C_{t+1}^{-\sigma}}{P_{t+1}}\right\},$$

which is the standard intertemporal optimality condition. Savings are optimal when the real marginal loss in terms of lower current consumption (left-hand side) equals the expected real marginal benefits of higher next-period consumption (right-hand side). Note that the expression can be rewritten as condition (7) in Galí (2008, Chapter 2, p. 17):

$$Q_t = \beta \mathrm{E}_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right\}. \tag{8}$$

This result shows that with complete financial markets and separable preferences, the dynamics of aggregate consumption are independent of the presence of wage rigidities, making the introduction of these into the basic model analytically as simple as possible.

Remark that the stochastic discount factor, $Q_{t,t+1}$, is given by the stochastic version of the right-hand side of (8):

$$Q_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}}$$

References

[1] Galí, J., 2008, Monetary Policy, Inflation, and the Business Cycle. Princeton: Princeton University Press.