Proof of Lemma 2 from the appendix to Chapter 4 in Galí (2008)

Henrik Jensen
Department of Economics
University of Copenhagen

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We want to prove the relationship between cross-sectional price dispersion and aggregate inflation as given by Lemma 2 on page 89 in Galí (2008). The lemma is

Lemma 2:

$$\sum_{t=0}^{\infty} \beta^t \mathrm{var}_i \{p_t(i)\} = \frac{\theta}{(1-\beta\theta)(1-\theta)} \sum_{t=0}^{\infty} \beta^t \pi_t^2.$$

To prove it, we need first to recapitulate some definitions regarding sectorial and aggregate prices. First, remember that

$$\begin{aligned} \mathrm{var}_i \{p_t(i)\} &\equiv \int_0^1 (p_t(i) - \mathrm{E}_i\{p_t(i)\})^2 \,\mathrm{d}i, \\ \mathrm{E}_i\{p_t(i)\} &\equiv \int_0^1 p_t(i)\,\mathrm{d}i. \end{aligned}$$

For notational simplicity, let (following Woodford, 2003, Chapter 6)

$$\begin{aligned} \Delta_t &\equiv var_i\{p_t(i)\}, \\ \overline{p}_t &\equiv \mathrm{E}_i\{p_t(i)\}. \end{aligned}$$

We see that the lemma expresses a relationship between the discounted values of $\mathrm{var}_i\{p_t(i)\}$ and π_t^2. This should hint us in the direction of searching for expressions for the evolution of variables over time. First look at $\overline{p}_t$. We have

$$\overline{p}_t - \overline{p}_{t-1} = \mathrm{E}_i\left\{p_t(i) - \overline{p}_{t-1}\right\}$$

Then use the Calvo structure:

$$\begin{aligned} \overline{p}_t - \overline{p}_{t-1} &= \theta \mathrm{E}_i\left\{p_{t-1}(i) - \overline{p}_{t-1}\right\} + (1-\theta)\left(p_t^* - \overline{p}_{t-1}\right) \\ &= (1-\theta)\left(p_t^* - \overline{p}_{t-1}\right). \end{aligned}$$

Note that with $p_t = \overline{p}_t$ this indeed provides the aggregate relationship we saw at the lectures:

$$\pi_t = (1-\theta)\left(p_t^* - \overline{p}_{t-1}\right) \tag{*}$$

Then look at $\text{var}_i \{p_t(i)\}$ and its dynamics, We can write

$$\Delta_t = \text{var}_i \left\{p_t(i) - \overline{p}_{t-1}\right\}$$

as $\text{var}_i \left\{\overline{p}_{t-1}\right\} = 0$. Hence,

$$\begin{aligned}
\Delta_t &= \text{E}_i \left\{ \left(p_t(i) - \overline{p}_{t-1} - \left[\text{E}_i \{p_t(i)\} - \overline{p}_{t-1}\right]\right)^2 \right\}, \\
&= \text{E}_i \left\{ \left(p_t(i) - \overline{p}_{t-1}\right)^2 \right\} - \left(\text{E}_i \{p_t(i)\} - \overline{p}_{t-1}\right)^2, \\
&= \text{E}_i \left\{ \left(p_t(i) - \overline{p}_{t-1}\right)^2 \right\} - \left(\overline{p}_t - \overline{p}_{t-1}\right)^2.
\end{aligned}$$

Then use Calvo structure:

$$\begin{aligned}
\Delta_t &= \theta \text{E}_i \left\{ \left(p_{t-1}(i) - \overline{p}_{t-1}\right)^2 \right\} + (1-\theta)\left(p_t^* - \overline{p}_{t-1}\right)^2 - \left(\overline{p}_t - \overline{p}_{t-1}\right)^2, \\
&= \theta \Delta_{t-1} + (1-\theta)\left(p_t^* - \overline{p}_{t-1}\right)^2 - \pi_t^2.
\end{aligned}$$

and (*):

$$\begin{aligned}
\Delta_t &= \theta \Delta_{t-1} + (1-\theta) \frac{1}{(1-\theta)^2} \left(\overline{p}_t - \overline{p}_{t-1}\right)^2 - \pi_t^2, \\
&= \theta \Delta_{t-1} + \frac{1}{1-\theta} \pi_t^2 - \pi_t^2, \\
&= \theta \Delta_{t-1} + \frac{\theta}{1-\theta} \pi_t^2. \qquad (**)
\end{aligned}$$

Then, at any t we can write cross-sectional price dispersion as

$$\Delta_t = \theta^{t+1} \Delta_{-1} + \sum_{s=0}^{t} \theta^{t-s} \left(\frac{\theta}{1-\theta}\right) \pi_s^2.$$

The term $\theta^{t+1}\Delta_{-1}$ is independent of policy from time $t \geq 0$ and onwards, so it can be moved into the "t.i.p." terms of the utility approximation. Then we can take the discounted sum on both sides:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \Delta_t &= \sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{t} \theta^{t-s} \left(\frac{\theta}{1-\theta}\right) \pi_s^2, \\
&= \frac{\theta}{1-\theta} \sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{t} \theta^{t-s} \pi_s^2.
\end{aligned}$$

The right-hand side summations take the form

$$\begin{array}{lc}
t = 0 & \pi_0^2 \\
t = 1 & \beta\left(\theta\pi_0^2 + \pi_1^2\right) \\
t = 2 & \beta^2\left(\theta^2\pi_0^2 + \theta\pi_1^2 + \pi_2^2\right) \\
t = 3 & \beta^3\left(\theta^3\pi_0^2 + \theta^2\pi_1^2 + \theta\pi_2^2 + \pi_3^2\right) \\
t = 4 & \beta^4\left(\theta^4\pi_0^2 + \theta^3\pi_1^2 + \theta^2\pi_2^2 + \theta\pi_3^2 + \pi_4^2\right) \\
\vdots & \vdots
\end{array}$$

Thereby we can see that

$$\sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{t} \theta^{t-s} \pi_s^2 = \sum_{s=0}^{\infty} \beta^s \pi_s^2 \sum_{t=s}^{\infty} (\beta\theta)^{t-s} \,,$$

and we then get

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \Delta_t &= \frac{\theta}{1-\theta} \sum_{s=0}^{\infty} \beta^s \pi_s^2 \sum_{t=s}^{\infty} (\beta\theta)^{t-s} \\
&= \frac{\theta}{(1-\beta\theta)(1-\theta)} \sum_{s=0}^{\infty} \beta^s \pi_s^2 \\
&= \frac{\theta}{(1-\beta\theta)(1-\theta)} \sum_{t=0}^{\infty} \beta^t \pi_t^2
\end{aligned}$$

which proves the lemma.

References

[1] Galí, J., 2008, Monetary Policy, Inflation and the Business Cycle. An Introduction to the New Keynesian Framework (Princeton University Press)

[2] Woodford, M., 2003, Interest and Prices. Foundations of a Theory of Monetary Policy (Princeton University Press).