Getting to page 31 in Galí (2008)

Henrik Jensen
Department of Economics
University of Copenhagen

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Abstract

This note shows in detail how to compute the solutions for output, inflation and the nominal interest rate in the classical model with money in the utility function presented in Galí (2008, p. 28–31). It utilizes the *method of undetermined coefficients* to derive the rational-expectations solution.

1 Introduction

These notes present detailed computations leading to the solutions for output, inflation and the nominal interest rate in a classical monetary model with money in the utility function in Galí (2008, Chapter 2, p. 31). All notation follows Galí (2008), and will not be explained unless needed. Equation numbers in this document are unique, and do not correspond to the similar equations in Galí (2008).

2 Deriving the relevant optimality conditions

Time is discrete, and in any period t, the representative household seeks to maximize a utility function

$$\mathrm{E}_t \left\{ \sum_{i=t}^{\infty} \beta^{t-i} U\left(C_i, \frac{M_i}{P_i}, N_i \right) \right\}.$$

This is done while satisfying the following budget constraint:

$$P_t C_t + Q_t B_t + M_t \leq B_{t-1} + M_{t-1} + W_t N_t - T_t. \tag{1}$$

Let total financial wealth at the end of period t be defined as $\mathcal{A}_{t-1} \equiv B_{t-1} + M_{t-1}$. The budget constraint (1) can then be written compactly as

$$P_t C_t + Q_t \mathcal{A}_t + (1 - Q_t) M_t \leq \mathcal{A}_{t-1} + W_t N_t - T_t. \tag{2}$$

Written like (2), one readily sees the opportunity cost of investing resources in money rather than bonds. The yield on bonds are $(1 - Q_t)/Q_t$, which approximately equals $i_t \equiv -\log Q_t$, where i_t is the nominal interest rate. We therefore have $\exp(i_t) = Q_t^{-1}$, and therefore

$$1 - Q_t = 1 - \exp(-i_t). \tag{3}$$

It thus follows that whenever $i_t > 0$, the opportunity cost of holding money is positive.

We find the necessary optimality conditions by setting up the Lagrangian:

$$\mathcal{L}_t = \mathrm{E}_t \left\{ \sum_{i=t}^{\infty} \beta^{t-i} \left[U\left(C_i, \frac{M_i}{P_i}, N_i\right) - \lambda_i \left(P_i C_i + Q_i \mathcal{A}_i + (1 - Q_i) M_i - \mathcal{A}_{i-1} - W_i N_i + T_i\right) \right] \right\},$$

where λ_t is the multiplier on (2).[1] The necessary first-order conditions at any t are

$$\frac{\partial \mathcal{L}_t}{\partial C_t} = 0: \qquad U_{c,t} = \lambda_t P_t \tag{4}$$

$$\frac{\partial \mathcal{L}_t}{\partial M_t} = 0: \qquad \frac{U_{m,t}}{P_t} = \lambda_t (1 - Q_i) \tag{5}$$

$$\frac{\partial \mathcal{L}_t}{\partial N_t} = 0: \qquad U_{n,t} = -\lambda_t W_t \tag{6}$$

$$\frac{\partial \mathcal{L}_t}{\partial \mathcal{A}_t} = 0: \qquad \lambda_t Q_t = \beta \mathrm{E}_t \{\lambda_{t+1}\} \tag{7}$$

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial \lambda_t} &\geq 0: \qquad P_t C_t + Q_t \mathcal{A}_t + (1 - Q_t) M_t \leq \mathcal{A}_{t-1} + W_t N_t - T_t, \\ \lambda_t &\geq 0, \end{aligned}$$

and the complementary slackness condition

$$\lambda_t \left(P_t C_t + Q_t \mathcal{A}_t + (1 - Q_t) M_t - \mathcal{A}_{t-1} - W_t N_t + T_t\right) = 0. \tag{8}$$

Note that since it is always assumed that the marginal utility of consumption is positive, $U_{c,t} > 0$, we have from (4) that $\lambda_t > 0$. Hence, from (8) we have that (2) always binds. Normally one ignores this step, and just state the budget constraint as an equality from the beginning.

Since the focus in this version of the classical model is the inclusion of money in the utility function, we start by characterizing optimal money demand. Combining (4) and (5) we readily get

$$\frac{U_{m,t}}{U_{c,t}} = (1 - Q_t), \tag{9}$$

which implicitly characterizes the optimal money demand as the quantity that equates the marginal rate of substitution between money and consumption to the marginal rate of transformation, which here is the opportunity cost of holding money. Expressed as a function of the nominal interest rate we get

$$\frac{U_{m,t}}{U_{c,t}} = 1 - \exp(-i_t). \tag{10}$$

[1] One can also write the Langrangian without having the discount factor being multiplied on Lagrange multipliers. This does not affect the results, but the current formulation is the conventional and readily gives λ_t the interpretation as the marginal utility of income at t.

From (10) we get a micro foundation for conventional money demand functions: For given consumption and labor, a higher nominal interest rate reduces money demand whenever $U_{mm,t} < 0$, i.e., for standard concave utility.

From the first-order conditions we also recover the standard optimality conditions for labor supply [combine (4) and (6)], and savings [combine (4) and (7)]:

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t}, \tag{11}$$

$$Q_t = \beta \mathrm{E}_t \left\{ \frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right\}. \tag{12}$$

The model's supply side is represented by competitive firms who produce output with labor input though the production function $Y_t = A_t N_t^{1-\alpha}$. Profit-maximizing labor demand is characterized by

$$(1-\alpha) A_t N_t^{-\alpha} = \frac{W_t}{P_t}. \tag{13}$$

The description of the model is complete, and the equilibrium values for five unknowns, C_t, Y_t, N_t, W_t/P_t and M/P_t, conditional on a monetary policy, i_t (and thus Q_t), can be determined from (10), (11), (12) and (13) along with the goods market clearing condition $Y_t = C_t$.[2] To facilitate a solution utility is assumed to have the following form

$$U\left(C_i, \frac{M_i}{P_i}, N_i\right) = \frac{X_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}, \tag{14}$$

where

$$X_t \equiv \left[(1-\vartheta) C_t^{1-\nu} + \vartheta \left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}, \qquad \nu > 0,\ \nu \neq 1, \tag{15}$$

$$\equiv C_t^{1-\vartheta} \left(\frac{M_t}{P_t}\right)^{\vartheta}, \qquad \nu = 1,$$

[2] Some may rightfully wonder what happened to the households' budget constraint. It is not ignored. In "the background" of the simple representation given here, the government pays out transfers $-T_t$, which are financed by printing money. I.e., the government budget constraint reads

$$-T_t = M_t - M_{t-1}.$$

Moreover, since households are identical there will be nobody holding positive positions in bonds. I.e., in equilibrium $B_t = 0$ all t. In equilibrium, the household's budget constraint is therefore

$$P_t C_t + M_t = M_{t-1} + W_t N_t - T_t.$$

Together with the public budget constraint one recovers

$$P_t C_t = W_t N_t.$$

Finally, since firms are competitive, profits, $P_t Y_t - W_t N_t$ are zero, i.e., $W_t N_t = P_t Y_t$, which inserted above gives

$$P_t C_t = P_t Y_t.$$

Hence, $C_t = Y_t$ as claimed in the main text. So, what may immediately seems "too simple," actually has solid foundations. In this model, one is just not interested in following the evolution of net asset holdings, profits or transfers (the first two variables are zero all t in any case, and transfers are just a residual of money creation, whose value must accrue to somebody in the economy).

is an index aggregating utility from consumption and real money holdings. With this functional form we get

$$\begin{aligned} U_{c,t} &= (1-\vartheta) X_t^{\nu-\sigma} C_t^{-\nu}, \\ U_{m,t} &= \vartheta X_t^{\nu-\sigma} \left(\frac{M_t}{P_t}\right)^{-\nu}, \end{aligned}$$

and can therefore rewrite (10), (11) and (12), respectively, as

$$\frac{M_t}{P_t} = C_t \left[1-\exp\left(-i_t\right)\right]^{-\frac{1}{\nu}} \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{1}{\nu}}, \tag{16}$$

$$N_t^{\varphi} X_t^{\sigma-\nu} C_t^{\nu} (1-\vartheta)^{-1} = \frac{W_t}{P_t}, \tag{17}$$

$$Q_t = \beta \mathrm{E}_t \left\{ \left(\frac{C_{t+1}}{C_t}\right)^{-\nu} \left(\frac{X_{t+1}}{X_t}\right)^{\nu-\sigma} \frac{P_t}{P_{t+1}} \right\}. \tag{18}$$

From this system, one sees that monetary policy can have real effects whenever $\nu \neq \sigma$. Only when $\nu = \sigma$, the system reduces to the basic classical model where monetary policy is irrelevant for the determination of, e.g., C_t and N_t. Otherwise, as changes in the nominal interest rate have impact on real money holdings [through (16)], the associated change in X_t will affect the labor supply decision; cf. (17). In the simulations presented in the slides associated with Chapter 2, the case of $\nu > \sigma$ was considered.[3] A shock driving up the nominal interest rate resulted in a drop in output. Although the simulations, which came from Walsh (2010), were for a somewhat richer model with physical capital formation, the economic transmission mechanism is the same here. A higher i_t leads to a lower M_t/P_t. This lowers X_t, which lowers the marginal utility of consumption when $\nu > \sigma$. For a given real wage, labor supply goes down, cf. (17), and output will fall.

3 Deriving the linearized system

The system cannot be solved analytically, so one log-linearizes the relevant equations around a zero-growth steady-state (this is an innocent assumption as the model does not contain any "engine" of sustainable growth). In particular, a zero-inflation steady state is considered. This is an assumption made not only for analytical tractability, but also because it makes results readily comparable to the zero-inflation steady states in the New-Keynesian models.[4]

[3] This appears to be the empirically relevant case, when one translate the parameters to their real world counterparts. The coefficient of relative risk aversion, σ, is usually found to be around 1–5. As ν is the inverse elasticity of substitution between money and consumption, estimates of the semi-elasticity of money with respect to the nominal interest rate, give guidance to the appropriate value (and an upper bound normally). As these elasticities often are in the order of 0.1, the inverse is well above the 1–5 range so as to make $\nu > \sigma$ the relevant case.

[4] There, the assumption is not innocent as it rules out dynamics of price dispersion, as these will be of only second-order importance around a zero-inflation steady state. Also, it facilitates welfare analyses based on second-order Taylor approximations of utility functions, because zero inflation is welfare optimal, implying that costs of inflation fluctuations will only be of second order. Thereby first-order approximations of the model equations suffice, as the omitted second-order terms only have third- or fourth-order welfare effects. See Chapter 4 of Galí (2008) for more details.

Most of the log-linearizations are straightforward, but those involving money demand and the composite consumption-real money balances index, are a bit involved, so we will devote some attention to these. Now, under the assumptions about the steady state, the consumption-Euler equation, (18), becomes

$$Q = \beta. \tag{19}$$

(Throughout, a variable without time index will be a steady-state value.) This has straightforward economic intuition. When the price of a bond today is exactly equal to the utility weight you attach to its return (which is one), you have no incentive to save or dissave so as to let your marginal utility of consumption differ across periods. Using (3) in the money demand function, (16), this is in steady state characterized by

$$\begin{aligned} \frac{M/P}{C} &= (1-Q)^{-\frac{1}{\nu}} \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{1}{\nu}}, \\ &= (1-\beta)^{-\frac{1}{\nu}} \left(\frac{\vartheta}{1-\vartheta}\right)^{\frac{1}{\nu}}, \\ &= \left[\frac{\vartheta}{(1-\beta)(1-\vartheta)}\right]^{\frac{1}{\nu}}, \\ &\equiv k_m, \end{aligned} \tag{20}$$

where the second line uses (19).

We can now log linearize (16) around the steady state. Letting lower-case letters denote log deviations from steady state, e.g., $c_t = \log(C_t/C)$, we readily get, by taking logs of (16) and (20) and differencing:

$$m_t - p_t - c_t = -\frac{1}{\nu}\left[\log(1-\exp(-i_t)) - \log(1-Q)\right]$$

The term in square brackets can be approximated to first order as

$$\begin{aligned} \log(1-\exp(-i_t)) - \log(1-Q) &\approx \log(1-\exp(-i)) - \log(1-Q) \\ &\quad + \frac{1}{1-\exp(-i)}\exp(-i)(i_t - i) \\ &= \frac{1}{\exp(i)-1}(i_t - i), \end{aligned}$$

where we have used (3). Note that as $i_t = -\log Q_t$, we have $i = -\log\beta = \rho$. Collecting these results we get the log-linear money demand expression:

$$m_t - p_t = c_t - \eta i_t, \tag{21}$$

with

$$\eta \equiv \frac{1}{\nu[\exp(i)-1]} = \frac{\beta}{\nu(1-\beta)}, \tag{22}$$

and where the steady-state term $\eta\rho$ has been ignored.[5] [The second equality in (22) follows from (3) and (19).]

[5] In a linear model, this has no implications for the analyses of the effects of various shocks. Alternatively, one could simply work with a new variable, $i_t - \rho$, i.e, the nominal interest rate's deviation from steady state. This is just a matter of preference for presentation.

We then proceed by deriving the labor supply schedule in log-deviations from steady state. From (17) we readily get

$$w_t - p_t = \sigma c_t + \varphi n_t + (\nu - \sigma)(c_t - x_t), \tag{23}$$

To eliminate x_t we first have to derive it. Recall that the composite consumption-real money balances index is defined as

$$X_t = \left[(1-\vartheta) C_t^{1-\nu} + \vartheta \left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}.$$

A first-order Taylor approximation of X_t around the steady state leads to

$$\begin{aligned} X_t &= X + \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1} (1-\vartheta) C^{-\nu} (C_t - C) \\ &\quad + \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1} \vartheta \left(\frac{M_t}{P_t}\right)^{-\nu} \left(\frac{M_t}{P_t} - \frac{M}{P}\right) \end{aligned}$$

where variables without time subscripts are steady-state values. Subtracting by X on both sides, and rearranging, this is readily rewritten as

$$\begin{aligned} \frac{X_t - X}{X} &= \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1} (1-\vartheta) C^{-\nu} \frac{C}{X} \frac{(C_t - C)}{C} \\ &\quad + \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1} \vartheta \left(\frac{M_t}{P_t}\right)^{-\nu} \frac{M/P}{X} \frac{\left(\frac{M_t}{P_t} - \frac{M}{P}\right)}{M/P}. \end{aligned}$$

Again, letting lower-case letters denote log-deviations from steady state, and acknowledging that $\log(X_t/X) \approx (X_t - X)/X$ is valid in a first-order approximation, we then find

$$\begin{aligned} x_t &= \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1} (1-\vartheta) C^{-\nu} \frac{C}{X} c_t \\ &\quad + \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1} \vartheta \left(\frac{M_t}{P_t}\right)^{-\nu} \frac{M/P}{X} (m_t - p_t), \\ &= \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{-1} (1-\vartheta) C^{1-\nu} c_t \\ &\quad + \left[(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}\right]^{-1} \vartheta \left(\frac{M_t}{P_t}\right)^{1-\nu} (m_t - p_t), \\ &= \frac{(1-\vartheta) C^{1-\nu}}{(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}} c_t + \frac{\vartheta \left(\frac{M}{P}\right)^{1-\nu}}{(1-\vartheta) C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}} (m_t - p_t). \end{aligned} \tag{24}$$

We then use (24) in (23) to obtain

$$\begin{aligned} w_t - p_t &= \sigma c_t + \varphi n_t \\ &+ (\nu - \sigma)\left(c_t - \frac{(1-\vartheta)\, C^{1-\nu}}{(1-\vartheta)\, C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}} c_t - \frac{\vartheta \left(\frac{M}{P}\right)^{1-\nu}}{(1-\vartheta)\, C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}} (m_t - p_t) \right), \\ w_t - p_t &= \sigma c_t + \varphi n_t + (\nu - \sigma) \frac{\vartheta \left(\frac{M}{P}\right)^{1-\nu}}{(1-\vartheta)\, C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}} \left[c_t - (m_t - p_t) \right], \end{aligned}$$

and thus

$$w_t - p_t = \sigma c_t + \varphi n_t + \chi (\nu - \sigma) \left[c_t - (m_t - p_t) \right], \tag{25}$$

with

$$\begin{aligned} \chi &\equiv \frac{\vartheta \left(\frac{M}{P}\right)^{1-\nu}}{(1-\vartheta)\, C^{1-\nu} + \vartheta \left(\frac{M}{P}\right)^{1-\nu}}, \\ &= \frac{\vartheta \left(\frac{M}{P}/C\right)^{1-\nu}}{1 - \vartheta + \vartheta \left(\frac{M}{P}/C\right)^{1-\nu}}. \end{aligned} \tag{26}$$

We can insert the steady-state value of $\frac{M}{P}/C$ from (20), to obtain

$$\begin{aligned} \chi &\equiv \frac{\vartheta \left[\frac{\vartheta}{(1-\beta)(1-\vartheta)}\right]^{\frac{1-\nu}{\nu}}}{1 - \vartheta + \vartheta \left[\frac{\vartheta}{(1-\beta)(1-\vartheta)}\right]^{\frac{1-\nu}{\nu}}}, \\ &= \frac{\vartheta^{\frac{1}{\nu}} \left[(1-\beta)(1-\vartheta)\right]^{\frac{\nu-1}{\nu}}}{1 - \vartheta + \vartheta^{\frac{1}{\nu}} \left[(1-\beta)(1-\vartheta)\right]^{\frac{\nu-1}{\nu}}}, \\ &= \frac{\vartheta^{\frac{1}{\nu}} (1-\beta)^{\frac{\nu-1}{\nu}}}{(1-\vartheta)^{\frac{1}{\nu}} + \vartheta^{\frac{1}{\nu}} (1-\beta)^{\frac{\nu-1}{\nu}}}, \end{aligned} \tag{27}$$

which is the definition of χ first used in the text (on page 29). Alternatively, one can through (26) express χ directly as a function of the steady-state ratio of real money balances to consumption, k_m:

$$\begin{aligned} \chi &= \frac{\vartheta k_m^{1-\nu}}{1 - \vartheta + \vartheta k_m^{1-\nu}}, \\ &= \frac{k_m}{\frac{1-\vartheta}{\vartheta} k_m^{\nu} + k_m}. \end{aligned}$$

Using (20), this can be written as

$$\begin{aligned}\chi &= \frac{k_m}{\frac{1-\vartheta}{\vartheta}\frac{\vartheta}{(1-\beta)(1-\vartheta)}+k_m}, \\ &= \frac{(1-\beta)k_m}{1+(1-\beta)k_m}, \end{aligned} \tag{28}$$

which is the second variation of χ used on page 29.

We now use the money demand function, (21), to substitute out $c_t-(m_t-p_t)$ in (25):

$$w_t-p_t=\sigma c_t+\varphi n_t+\chi\eta(\nu-\sigma)i_t.$$

More compactly, this is written as

$$w_t-p_t=\sigma c_t+\varphi n_t+\omega i_t, \tag{29}$$

where

$$\begin{aligned}\omega &\equiv \chi\eta(\nu-\sigma)=\frac{(1-\beta)k_m}{1+(1-\beta)k_m}\frac{\beta}{\nu(1-\beta)}(\nu-\sigma), \\ &= \frac{\beta k_m\left(1-\frac{\sigma}{\nu}\right)}{1+(1-\beta)k_m}.\end{aligned}$$

We here clearly see how the nominal interest rate affect labor supply as long as $\nu\neq\sigma$.

$$\begin{aligned}Q_t &= \beta\mathrm{E}_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\left(\frac{X_{t+1}}{X_t}\right)^{\nu-\sigma}\frac{P_t}{P_{t+1}}\right\} \\ &= \beta\mathrm{E}_t\left\{\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\left(\frac{X_{t+1}}{X_t}\right)^{\nu-\sigma}\frac{1}{1+\pi_{t+1}}\right\}\end{aligned}$$

We then linearize (18) around the steady state. A first-order Taylor approximation yields

$$\begin{aligned}Q_t-Q &= -\beta\nu\frac{1}{C}\left(\mathrm{E}_t\left\{C_{t+1}\right\}-C\right)+\beta\nu\frac{1}{C}\left(C_t-C\right) \\ &\quad+\beta(\nu-\sigma)\frac{1}{X}\left(\mathrm{E}_t\left\{X_{t+1}\right\}-X\right)-\beta(\nu-\sigma)\frac{1}{X}\left(X_t-X\right) \\ &\quad-\beta\mathrm{E}_t\pi_{t+1}\end{aligned}$$

which becomes

$$\frac{Q_t-Q}{Q}=-\nu\left(\mathrm{E}_t\left\{c_{t+1}\right\}-c_t\right)+(\nu-\sigma)\left(\mathrm{E}_t\left\{x_{t+1}\right\}-x_t\right)-\mathrm{E}_t\pi_{t+1}.$$

We can write the left-hand side as

$$-\frac{1-Q_t-(1-Q)}{Q}\simeq -i_t+\rho.$$

Hence we get

$$-i_t+\rho=-\nu\left(\mathrm{E}_t\left\{c_{t+1}\right\}-c_t\right)+(\nu-\sigma)\left(\mathrm{E}_t\left\{x_{t+1}\right\}-x_t\right)-\mathrm{E}_t\pi_{t+1},$$

which can be re-written as

$$-i_t+\rho=-\sigma\left(\mathrm{E}_t\left\{c_{t+1}\right\}-c_t\right)+(\nu-\sigma)\left(\mathrm{E}_t\left\{x_{t+1}\right\}-\mathrm{E}_t\left\{c_{t+1}\right\}-x_t+c_t\right)-\mathrm{E}_t\pi_{t+1},$$

and thereby

$$c_t=\mathrm{E}_t\left\{c_{t+1}\right\}-\frac{1}{\sigma}\left(i_t-\mathrm{E}_t\pi_{t+1}-\rho-(\nu-\sigma)\,\mathrm{E}_t\left\{c_{t+1}-x_{t+1}-\left(c_t-x_t\right)\right\}\right). \tag{30}$$

Again, we see that for the case of $\nu=\sigma$ we have the standard case with no money in the utility function. Like in the derivation of the labor supply function, we can eliminate x_t in (30) through (24) so as to get:

$$c_t=\mathrm{E}_t\left\{c_{t+1}\right\}-\frac{1}{\sigma}\left(i_t-\mathrm{E}_t\pi_{t+1}-\rho-\chi\,(\nu-\sigma)\,\mathrm{E}_t\left\{c_{t+1}-c_t-\left[\left(m_{t+1}-p_{t+1}\right)-\left(m_t-p_t\right)\right]\right\}\right).$$

The log-linearized money demand function, (21), is then used to eliminate $c_{t+1}-c_t-\left[\left(m_{t+1}-p_{t+1}\right)-\left(m_t-p_t\right)\right]$ to yield

$$c_t=\mathrm{E}_t\left\{c_{t+1}\right\}-\frac{1}{\sigma}\left(i_t-\mathrm{E}_t\pi_{t+1}-\rho-\omega\mathrm{E}_t\left\{i_{t+1}-i_i\right\}\right). \tag{31}$$

Imposing $y_t=c_t$, (31) readily becomes the "dynamic IS curve" for this classical money-in-the utility function model:

$$y_t=\mathrm{E}_t\left\{y_{t+1}\right\}-\sigma^{-1}\left(i_t-\mathrm{E}_t\left\{\pi_{t+1}\right\}-\rho-\omega\mathrm{E}_t\left\{\Delta i_{t+1}\right\}\right). \tag{32}$$

To obtain labor market equilibrium, we log linearize the labor demand equation (13) to get

$$a_t-\alpha n_t=w_t-p_t,$$

which in combination with (29) gives

$$\sigma y_t+\varphi n_t+\omega i_t=a_t-\alpha n_t,$$

where $y_t=c_t$ has been used. Since the production function in logs is $y_t=a_t+(1-\alpha)\,n_t$, we use $n_t=(1-\alpha)^{-1}\left(y_t-a_t\right)$ to substitute out n_t:

$$\begin{aligned}
\sigma y_t+\varphi\,(1-\alpha)^{-1}\left(y_t-a_t\right)+\omega i_t &= a_t-\alpha\,(1-\alpha)^{-1}\left(y_t-a_t\right),\\
y_t\frac{\sigma\,(1-\alpha)+\varphi+\alpha}{1-\alpha} &= \frac{1+\varphi}{1-\alpha}a_t-\omega i_t,\\
y_t &= \frac{1+\varphi}{\sigma\,(1-\alpha)+\varphi+\alpha}a_t-\frac{\omega\,(1-\alpha)}{\sigma\,(1-\alpha)+\varphi+\alpha}i_t \qquad (33)\\
&\equiv \psi_{ya}a_t-\psi_{yi}i_t.
\end{aligned}$$

4 Solving for the rational-expectations equilibrium

With expressions (32) and (33) we are now ready to solve for output and inflation given a specification for interest-rate policy. It is assumed that the nominal interest rate is determined according to the rule

$$i_t=\rho+\phi_\pi\pi_t+v_t, \qquad \phi_\pi>1. \tag{34}$$

Furthermore, the exogenous disturbances follow AR(1) processes:

$$\begin{aligned}
a_t &= \rho_a a_{t-1}+\varepsilon_t^a,\\
v_t &= \rho_v v_{t-1}+\varepsilon_t^v.
\end{aligned}$$

4.1 Finding the relevant difference equation

The solution proceeds by inserting (34) into (32) and (33):

$$\begin{aligned} y_t &= \mathrm{E}_t\{y_{t+i}\} - \sigma^{-1}(\phi_\pi \pi_t + v_t - \mathrm{E}_t\{\pi_{t+1}\} - \omega \mathrm{E}_t\{\phi_\pi \Delta\pi_{t+1} + \Delta v_{t+1}\}) \\ y_t &= \psi_{ya} a_t - \psi_{yi}(\rho + \phi_\pi \pi_t + v_t), \end{aligned}$$

Using the latter in the former yields

$$\begin{aligned} &\psi_{ya} a_t - \psi_{yi}(\rho + \phi_\pi \pi_t + v_t) \\ = \; &\mathrm{E}_t\left\{\psi_{ya} a_{t+1} - \psi_{yi}(\rho + \phi_\pi \pi_{t+1} + v_{t+1})\right\} \\ &-\sigma^{-1}(\phi_\pi \pi_t + v_t - \mathrm{E}_t\{\pi_{t+1}\} - \omega \mathrm{E}_t\{\phi_\pi \Delta\pi_{t+1} + \Delta v_{t+1}\}). \end{aligned}$$

Solving out expectations terms using the properties of the shock processes gives

$$\begin{aligned} &\psi_{ya} a_t - \psi_{yi}(\rho + \phi_\pi \pi_t + v_t) \\ = \; &\psi_{ya}\rho_a a_t - \psi_{yi}(\rho + \phi_\pi \mathrm{E}_t\{\pi_{t+1}\} + \rho_v v_t) \\ &-\sigma^{-1}[\phi_\pi \pi_t + v_t - \mathrm{E}_t\{\pi_{t+1}\} - \omega\phi_\pi \mathrm{E}_t\{\Delta\pi_{t+1}\} - \omega\phi_\pi(\rho_v - 1)v_t]. \end{aligned}$$

Rearranging:

$$\begin{aligned} &-\sigma\psi_{ya} a_t + \sigma\psi_{yi}(\rho + \phi_\pi \pi_t + v_t) \\ = \; &-\sigma\psi_{ya}\rho_a a_t + \sigma\psi_{yi}(\rho + \phi_\pi \mathrm{E}_t\{\pi_{t+1}\} + \rho_v v_t) \\ &+\phi_\pi \pi_t + v_t - \mathrm{E}_t\{\pi_{t+1}\} - \omega\phi_\pi \mathrm{E}_t\{\pi_{t+1}\} + \omega\phi_\pi \pi_t + \omega\phi_\pi(1 - \rho_v)v_t. \end{aligned}$$

Collecting terms:

$$\begin{aligned} &-\sigma\psi_{ya}(1 - \rho_a)a_t - \left(1 + (1 - \rho_v)\left(\omega - \sigma\psi_{yi}\right)\right)v_t \\ = \; &\phi_\pi\left(1 + \omega - \sigma\psi_{yi}\right)\pi_t - \left(1 + \phi_\pi\left(\omega - \sigma\psi_{yi}\right)\right)\mathrm{E}_t\{\pi_{t+1}\}. \end{aligned}$$

Note that

$$\begin{aligned} \omega - \sigma\psi_{yi} &= \omega - \sigma\frac{\omega(1-\alpha)}{\sigma(1-\alpha) + \varphi + \alpha}, \\ &= \omega\left(1 - \frac{\sigma(1-\alpha)}{\sigma(1-\alpha) + \varphi + \alpha}\right), \\ &= \omega\frac{\varphi + \alpha}{\sigma(1-\alpha) + \varphi + \alpha}, \\ &= \omega\psi, \end{aligned}$$

with

$$\psi \equiv \frac{\varphi + \alpha}{\sigma(1-\alpha) + \varphi + \alpha}.$$

Hence, we can write

$$\begin{aligned} &-\sigma\psi_{ya}(1 - \rho_a)a_t - (1 + (1 - \rho_v)\omega\psi)v_t \\ = \; &\phi_\pi(1 + \omega\psi)\pi_t - (1 + \phi_\pi\omega\psi)\mathrm{E}_t\{\pi_{t+1}\}, \end{aligned}$$

leading to a first-order rational expectations difference equation in π_t:

$$\begin{aligned} \pi_t &= \frac{1 + \phi_\pi\omega\psi}{\phi_\pi(1 + \omega\psi)}\mathrm{E}_t\{\pi_{t+1}\} - \frac{\sigma\psi_{ya}(1 - \rho_a)}{\phi_\pi(1 + \omega\psi)}a_t - \frac{1 + (1 - \rho_v)\omega\psi}{\phi_\pi(1 + \omega\psi)}v_t, \\ \pi_t &= \Theta\mathrm{E}_t\{\pi_{t+1}\} - \frac{\sigma\psi_{ya}(1 - \rho_a)}{\phi_\pi(1 + \omega\psi)}a_t - \frac{1 + (1 - \rho_v)\omega\psi}{\phi_\pi(1 + \omega\psi)}v_t, \qquad (35) \end{aligned}$$

where

$$\Theta \equiv \frac{1+\phi_\pi \omega\psi}{\phi_\pi (1+\omega\psi)}.$$

This has a unique stationary solution iff

$$-1 < \Theta < 1. \tag{36}$$

In the case where $\omega > 0$, this is clearly satisfied as $\phi_\pi > 1$. However, in the case where $\omega < 0$ we cannot be sure that it holds (for $\omega\psi \to -1$, $\Theta \to -\infty$). We assume that (36) holds in the following.

4.2 Solving for π_t by the method of undetermined coefficients

We can now solve (35) for π_t by forward substitution. This is cumbersome, however, so instead we will solve (35) by the *method of undetermined coefficients.* This method involves two simple steps. In the first, one makes a conjecture about the form of the solution as a function of unknown coefficients. In the second step, one uses the conjecture together with the difference equation to verify the validity of the conjecture and to identify the unknown coefficients (which then implies that a solution is obtained).

In this case (and in all related cases of linear rational expectations models), it is natural to conjecture that inflation is a linear function of the shocks a_t and v_t. I.e.,

$$\pi_t = -Aa_t - Bv_t, \tag{37}$$

where A and B are the undetermined coefficients to be identified. Forward (37) one period and take expectations:

$$\begin{aligned} \mathrm{E}_t \{\pi_{t+1}\} &= -A\mathrm{E}_t \{a_{t+1}\} - B\mathrm{E}_t \{v_{t+1}\}, \\ \mathrm{E}_t \{\pi_{t+1}\} &= -A\rho_a a_t - B\rho_v v_t. \end{aligned} \tag{38}$$

We now combine (38) with (35) to see whether the conjectured form of the solution is correct:

$$\begin{aligned} \pi_t &= \Theta [-A\rho_a a_t - B\rho_v v_t] \\ &\quad -\frac{\sigma\psi_{ya}(1-\rho_a)}{\phi_\pi (1+\omega\psi)} a_t - \frac{1+(1-\rho_v)\omega\psi}{\phi_\pi (1+\omega\psi)} v. \end{aligned} \tag{39}$$

Wee see that it is. The conjecture is consistent with the difference equation; i.e., inflation *is* a linear function of the shocks. We can then identify A and B by using (37) together with (39):

$$\begin{aligned} -Aa_t - Bv_t &= \Theta [-A\rho_a a_t - B\rho_v v_t] \\ &\quad -\frac{\sigma\psi_{ya}(1-\rho_a)}{\phi_\pi (1+\omega\psi)} a_t - \frac{1+(1-\rho_v)\omega\psi}{\phi_\pi (1+\omega\psi)} v \end{aligned}$$

and note that this must hold for *all* values of a_t and v_t. Hence, the following equations apply:

$$-A = -\Theta A\rho_a - \frac{\sigma\psi_{ya}(1-\rho_a)}{\phi_\pi (1+\omega\psi)}, \tag{40}$$

$$-B = -\Theta B\rho_v - \frac{1+(1-\rho_v)\omega\psi}{\phi_\pi (1+\omega\psi)}. \tag{41}$$

From (40) and (41) we get

$$\begin{aligned} A &= \frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}, \\ B &= \frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}, \end{aligned}$$

respectively, giving us the solution for inflation as

$$\pi_t = -\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t - \frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t, \tag{42}$$

which is the expression in Galí (2008), p. 31. The solution for i_t follows immediately by inserting (42) into (34) (Galí ignores the constant ρ on p. 31; cf. the discussion in Footnote 5 on this note). I.e., we get

$$\begin{aligned} i_t &= \rho-\phi_\pi\left(\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t+\frac{1+\left(1-\rho_v\right)\omega\psi}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t\right)+v_t, \\ &= \rho-\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t+\left(1-\frac{1+\left(1-\rho_v\right)\omega\psi}{\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}\right)v_t, \\ &= \rho-\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t+\frac{\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)-1-\left(1-\rho_v\right)\omega\psi}{\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t, \\ &= \rho-\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t+\frac{-\left(1+\omega\psi\right)\Theta\rho_v+\rho_v\omega\psi}{\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t, \\ &= \rho-\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t+\frac{-\rho_v\left[\left(1+\omega\psi\right)\dfrac{1+\phi_\pi\omega\psi}{\phi_\pi\left(1+\omega\psi\right)}-\omega\psi\right]}{\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t, \\ &= \rho-\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t-\frac{\rho_v}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t, \end{aligned} \tag{43}$$

where the fifth line makes use of the definition of Θ.

Finally, the solution for y_t follows by inserting (43) into (33):

$$\begin{aligned} y_t &= \psi_{ya}a_t+\psi_{yi}\left(\frac{\sigma\psi_{ya}\left(1-\rho_a\right)}{\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}a_t+\frac{\rho_v}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t\right), \\ &= \psi_{ya}\left(1+\frac{\sigma\psi_{yi}\left(1-\rho_a\right)}{\left(1+\omega\psi\right)\left(1-\Theta\rho_a\right)}\right)a_t+\frac{\rho_v\psi_{yi}}{\phi_\pi\left(1+\omega\psi\right)\left(1-\Theta\rho_v\right)}v_t. \end{aligned} \tag{44}$$

References

[1] Galí, J., 2008, Monetary Policy, Inflation and the Business Cycle. Princeton University Press, Princeton, NJ.

[2] Walsh, C. E., 2010, Monetary Theory and Policy, Third Edition, The MIT Press, Boston, MA.