1. Money in the utility function (start)
   a. The basic money-in-the-utility function model
   b. Optimal behavior and steady-state equilibrium properties:
      Long-run superneutrality of money

Literature: Walsh (2010, Chapter 2, pp. 33–52)
Introductory remarks

• The standard model for (exogenous) economic growth is the simple Solow model featuring
  – a fixed savings rate
  – a law of motion for physical capital accumulation

• When extended with optimizing savings behavior, we get the Ramsey model

• Models have no role for money and, hence, monetary policy

• Purpose of models/analyses in coming lectures is to introduce a role for money in these type of models
• Money is introduced in various ways: Often short cuts
  – Short cuts are helpful for understanding simple features, and the more robust results are to a particular short cut, of course, the better

• Tobin model (1965, Econometrica) extends the Solow model by postulating a demand for money (the short cut)
  – Highlights the implications of inflation for the choice between investment in physical and financial assets
  – Higher inflation leads to a substitution away from financial assets towards physical capital
  – Higher inflation leads to higher steady-state capital stock and output

• Money-in-the-Utility-function models extend the Ramsey model by postulating that money gives utility (the short cut)
  – Highlights the importance of micro foundations and optimizing private-sector behavior (absent in the Tobin model)
Money in the utility function (start)

- A standard Ramsey model with money (infinite-horizon setting with optimizing households and perfect competition in goods market)

- The short-cut here: real money provides utility per se
  
  – One interpretation: Money facilitates transactions on the market and reduces “shopping time” (money as such is an otherwise useless commodity ...)

- The per-period utility function of households is
  \[ U_t = u(c_t, m_t) \]

  with \( u \) being increasing and concave in both arguments. (It is, of course, real money that enters in \( u \), i.e., \( M_t \)’s value relative to what it can buy at price \( P_t \).)

- Often, to ensure existence of an equilibrium where money is held (a “monetary equilibrium”), it is assumed that for some \( \bar{m} \), \( u_m(c_t, \bar{m}) = 0 \) and \( u_m(c_t, m_t) < 0 \) for \( m_t > \bar{m} \)

  – I.e., \( \bar{m} \) is an optimal quantity of money
• Aim of representative household is to maximize:

\[ W = \sum_{t=0}^{\infty} \beta^t u (c_t, m_t), \quad 0 < \beta < 1. \]  

(2.1)

• Households’ budget constraints are (ignoring for simplicity financial assets \(B_t\) used in Walsh):

\[ C_t + K_t + \frac{M_t}{P_t} = Y_t + \tau_t N_t + (1 - \delta) K_{t-1} + \frac{M_{t-1}}{P_t}, \quad 0 < \delta < 1 \]  

(2.2’)

• Note, \(m_t \equiv M_t / (P_t N_t)\) is end-of-period money balances; debatable in itself whether it should be \(m_{t-1}\) or \(m_t\) that should give utility.

• Output is produced by a CRS production function:

\[ Y_t = F (K_{t-1}, N_t). \]

In “intensive” form:

\[ y_t = f (k_{t-1}), \quad y_t \equiv Y_t / N_t, \quad k_{t-1} \equiv K_{t-1} / N_{t-1} \]

(population grows at a constant rate \(N_t / N_{t-1} = 1 + n\))

• Government makes per capita real lump-sum transfers (or withdrawals) to households in the form of money supply changes:

\[ \tau_t = \frac{M_t - M_{t-1}}{P_t N_t} \]

(This is the simple government budget constraint.)
In per-capita version, assuming no population growth in contrast to Walsh (set \( n = 0 \), and \( N_t = 1 \)), we get

\[ c_t + k_t + m_t = y_t + \tau_t + (1 - \delta) k_{t-1} + \frac{1}{1 + \pi_t} m_{t-1} \]

- Inflation \( \pi_t \equiv (P_t/P_{t-1}) - 1 \), erodes initial real monetary resources

Rewritten:

\[ c_t + k_t + m_t = \omega_t \]

\[ \equiv f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \frac{1}{1 + \pi_t} m_{t-1} \quad (2.4') \]

- Hence, \( \omega_t \) is the total available resources, treated as given at \( t \) by the households.

  - It is the relevant state variable when choosing the optimal paths of \( c, k, \) and \( m \) at date \( t \)

Household’s optimization problem is solved by dynamic programming
• Dynamic programming involves use of the *value function*—the maximal value of $W$,
  
  – given constraints
  – given optimal behavior of the household now and forever after
  – given the current state $\omega$

• The value function $V$ therefore satisfies

$$V(\omega_t) = \max \left\{ u(c_t, m_t) + \beta u(c_{t+1}, m_{t+1}) + \beta^2 u(c_{t+2}, m_{t+2}) + \ldots \right\}$$

$$V(\omega_t) = \max \left\{ u(c_t, m_t) + \beta \left[ u(c_{t+1}, m_{t+1}) + \beta u(c_{t+2}, m_{t+2}) + \ldots \right] \right\}$$

compactly written as

$$V(\omega_t) = \max \left\{ u(c_t, m_t) + \beta V(\omega_{t+1}) \right\}$$

• Maximization is over $c_t, m_t$ and $k_t$ subject to the budget constraint and the definition of $\omega_{t+1}$
  (available resources one period ahead)
To make it simple here, one

- substitutes out $\omega_{t+1}$
- then substitutes out $k_t$ using $k_t = \omega_t - c_t - m_t$

One then maximizes (unconstrained) “just” over $c_t$ and $m_t$:

$$\max \left\{ u(c_t, m_t) + \beta V \left( f(k_t) + \tau_{t+1} + (1 - \delta) k_t + \frac{1}{1 + \pi_{t+1}} m_t \right) \right\}$$

and thus

$$\max \left\{ u(c_t, m_t) + \beta V \left( f(\omega_t - c_t - m_t) + \tau_{t+1} + (1 - \delta) (\omega_t - c_t - m_t) + \frac{1}{1 + \pi_{t+1}} m_t \right) \right\}$$
• First-order condition concerning choice of $c_t$:

$$u_c (c_t, m_t) + \beta V_\omega (\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial c_t} = 0$$

$$u_c (c_t, m_t) = \beta V_\omega (\omega_{t+1}) [f_k (k_t) + 1 - \delta] \quad (2.6')$$

Marginal utility of period-$t$ consumption equals its marginal loss (in terms of the discounted marginal value of future capital)

• First-order condition concerning choice of $m_t$:

$$u_m (c_t, m_t) + \beta V_\omega (\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial m_t} = 0$$

$$u_m (c_t, m_t) + \beta V_\omega (\omega_{t+1}) \frac{1}{1 + \pi_{t+1}} = \beta V_\omega (\omega_{t+1}) (f_k (k_t) + 1 - \delta) \quad (2.8')$$

Marginal utility period-$t$ real money (in terms of direct utility plus discounted marginal value of more future real money) equals marginal loss (in terms of the marginal value of less future capital)

• Furthermore, transversality conditions must be satisfied:

$$\lim_{t \to \infty} \beta^t u_c (c_t, m_t) k_t = 0$$

$$\lim_{t \to \infty} \beta^t u_c (c_t, m_t) m_t = 0$$

(otherwise over-accumulation of $k$ and $m$ is taking place—lifetime utility could be improved through higher consumption by accumulating less)
• In the first-order conditions, $V$ is eliminated by use of the so-called *Envelope Theorem*

• Note: optimal period $t$ consumption and money holding choices will be functions of $\omega_t$

Define these as $c_t \equiv c(\omega_t)$ and $m_t \equiv m(\omega_t)$, respectively.

The value function is therefore *by definition* given by

$$V(\omega_t) = u(c(\omega_t), m(\omega_t)) + \beta V(\omega_{t+1}). \quad (*)$$

As (*) holds for *any* value of $\omega_t$, it follows that

$$V_\omega(\omega_t) = u_c(c(\omega_t), m(\omega_t)) c'(\omega_t) + u_m(c(\omega_t), m(\omega_t)) m'(\omega_t) + \beta V_\omega(\omega_{t+1}) \frac{\partial \omega_{t+1}}{\partial \omega_t}. \quad (**)$$

• Now, find $\partial \omega_{t+1}/\partial \omega_t$ when $c_t = c(\omega_t)$ and $m_t = m(\omega_t)$ applies:

Remember

$$\omega_{t+1} = f(k_t) + \tau_{t+1} + (1 - \delta) k_t + \frac{1}{1 + \pi_{t+1}} m_t$$

$$= f(\omega_t - c_t - m_t) + \tau_{t+1} + (1 - \delta) (\omega_t - c_t - m_t) + \frac{1}{1 + \pi_{t+1}} m_t$$

One therefore gets

$$\frac{\partial \omega_{t+1}}{\partial \omega_t} = [f_k(k_t) + 1 - \delta] (1 - c'(\omega_t) - m'(\omega_t)) + \frac{1}{1 + \pi_{t+1}} m'(\omega_t)$$
Combining this with (**):

\[
V_\omega(\omega_t) = u_c(c(\omega_t), m(\omega_t)) c'(\omega_t) + u_m(c(\omega_t), m(\omega_t)) m'(\omega_t)
+ \beta V_\omega(\omega_{t+1}) \left\{ \frac{[f_k(k_t) + 1 - \delta][1 - c'(\omega_t) - m'(\omega_t)]}{1 + \pi_{t+1}} m'(\omega_t) \right\}.
\]

Collect the \(c'(\omega_t)\) and \(m'(\omega_t)\) terms:

\[
V_\omega(\omega_t) = [u_c(c(\omega_t), m(\omega_t)) - \beta V_\omega(\omega_{t+1})(f_k(k_t) + 1 - \delta)] c'(\omega_t)
+ \left[ u_m(c(\omega_t), m(\omega_t)) + \beta V_\omega(\omega_{t+1}) \frac{1}{1 + \pi_{t+1}} \right] m'(\omega_t)
+ \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta]
\]

... and get a pleasant surprise:

\[
V_\omega(\omega_t) = \left[ u_c(c(\omega_t), m(\omega_t)) - \beta V_\omega(\omega_{t+1})(f_k(k_t) + 1 - \delta) \right] c'(\omega_t)
= 0 \text{ by (2.6')}\]

\[
+ \left[ u_m(c(\omega_t), m(\omega_t)) + \beta V_\omega(\omega_{t+1}) \frac{1}{1 + \pi_{t+1}} \right] m'(\omega_t)
= 0 \text{ by (2.8')}\]

\[
+ \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta].
\]
• All terms in front of $c'(\omega_t)$ and $m'(\omega_t)$ are zero by the first-order conditions
  
  – Note the terms capture the life-time utility effects of changing $\omega_t$ through the associated changes in $c_t$ and $m_t$
  
  – As the value function is defined as the life-time utility where $c_t$ and $m_t$ are optimally chosen, their effects are zero
  
  – Alternatively, use a contradiction argument: If a change in $\omega_t$ leads to a change in $V$ through the changes in $c_t$ and $m_t$, then $c_t$ and $m_t$ have not been optimally chosen, and $V$ is not the highest life-time utility.

• Hence, (***) reduces to
  
  $$V_\omega (\omega_t) = \beta V_\omega (\omega_{t+1}) [f_k (k_t) + 1 - \delta]$$

• Then use the first-order condition for consumption choice,
  
  $$u_c (c_t, m_t) - \beta V_\omega (\omega_{t+1}) [f_k (k_t) + 1 - \delta] = 0,$$

  to obtain Walsh’s expression:
  
  $$V_\omega (\omega_t) = u_c (c_t, m_t). \quad (2.10)$$

  – Marginal utility of consumption (denoted $\lambda_t$ in Walsh) equals marginal value of wealth
The first-order conditions can then be rewritten as
\[ u_c(c_t, m_t) = \beta u_c(c_{t+1}, m_{t+1}) [f_k(k_t) + 1 - \delta] \]
(a discrete-time, monetary version of the standard “Keynes-Ramsey rule”), and
\[ u_m(c_t, m_t) + \beta u_c(c_{t+1}, m_{t+1}) \frac{1}{1 + \pi_{t+1}} = u_c(c_t, m_t) \]
(marginal gain of \( m_t \) equal to the marginal loss in terms of lower capital in period \( t + 1 \)—equal to the marginal utility of \( c_t \) by the “Keynes-Ramsey rule”)

These conditions, together with the budget constraint characterize the optimal paths of \( c, k, \) and \( m \)

**NOTE:** Only \( m_t \) appears. Any change in \( M_t \) is reflected proportionally in \( P_t \): *Money neutrality*

For now, concentrate on the long-run properties of the model; i.e., a steady state with
\[ \Delta c_t = \Delta k_t = \Delta m_t = 0 \]

First, from “Keynes-Ramsey rule” it follows that in steady state
\[ 1 = \beta [f_k(k^{ss}) + 1 - \delta] , \]
or,
\[ f_k(k^{ss}) + 1 - \delta = \frac{1}{\beta} \quad (2.19') \]
This—*independently* of any monetary factors—defines the steady-state capital per capita (and thus output per capita).
• **Strong contrast** with Tobin model mentioned in introduction

• Difference is because this model envisions micro-founded behavior.

  − If, e.g., \( k_t < k^{ss} \) the current marginal product of capital is relatively high (as \( f_{kk} < 0 \))
    
    ⇒ optimal to postpone consumption to later
    
    ⇒ capital is accumulated until \( f_k (k^{ss}) + 1 - \delta = 1/\beta \) holds again

• − **If** one imagined that a Tobin effect was there; one would be self-contradictory:
  
  * Assume higher inflation increases capital to a new, higher steady state
  * Marginal product of capital decreases, and households would want to consume now rather than later
    
    ⇒ they endogenously save less and capital decreases until \( k^{ss} \) is reached again!
  
  * Inflation has no steady-state effect on capital. Only possible in Tobin model, where individuals are modelled as having an **exogenously** fixed savings rate
  
  * Example of the importance of considering changes in private sector behavior when examining policy changes
• What about steady-state consumption in the MIU model?

The budget constraint in steady state is

\[ f(k^{ss}) + \tau^{ss} + (1 - \delta) k^{ss} + \frac{1}{1 + \pi} m^{ss} = c^{ss} + k^{ss} + m^{ss} \]

Transfers are

\[ \tau_t = \frac{(M_t - M_{t-1})}{P_t}, \]
\[ = \frac{\theta_t M_{t-1}}{P_t}, \]
\[ = \frac{\theta_t}{1 + \pi_t} m_{t-1}, \]

so in steady state:

\[ \tau^{ss} = \frac{\theta}{1 + \pi} m^{ss} \]

Since a constant \( m \) implies \( \pi = \theta \), one gets

\[ f(k^{ss}) - \delta k^{ss} = c^{ss} \quad (2.21) \]

The economy’s resource constraint (national account): Output less gross investment equals consumption

• Implication is that \( c^{ss} \) is determined exclusively by \( k^{ss} \); and thus independent of nominal money growth

• Model exhibits superneutrality of money in steady state
• Do money growth and inflation not affect anything?

Yes, the opportunity cost of holding real money, and thus the steady-state value of real money balances

• Relative demand for consumption versus real money is given by

[use first-order condition for money holdings and divide through by \( u_c(c_t, m_t) \)]

\[
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{u_c(c_t, m_t)}{u_c(c_t, m_t)} - \frac{1}{1 + \pi_{t+1}} \frac{\beta u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)}
\]

\[
= 1 - \frac{1}{1 + \pi_{t+1}} \left( \frac{1}{f_k(k_t) + 1 - \delta} \right)
\]

\[
= 1 - \frac{1}{(1 + \pi_{t+1})(1 + r_t)}
\]

with \( r_t \equiv f_k(k_t) - \delta \) being the real interest rate

• Note that the real interest rate is the nominal rate net of expected inflation:

\[
1 + r_t = (1 + i_t) / (1 + \pi_{t+1}), \quad (r_t \approx i_t - \pi_{t+1})
\]

Hence,

\[
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{i_t}{1 + i_t} \equiv I_t
\]

(2.12)

So, as nominal interest rate is determined by the Fisher relationship, \( i_t \approx r_t + \pi_{t+1} \), higher inflation leads to a higher nominal interest rate

- Consistent with long-run data
For given \( c_t \), \( m_t \) is likely to fall when \( i_t \) increases (as \( u_{mm} < 0 \)).

- A micro foundation for conventional money-demand function
- With \( u(c_t, m_t) = \left[ ac_t^{1-b} + (1 - a) m_t^{1-b} \right]^{1/(1-b)} \), \( 0 < a < 1, b > 0 \) we get from (2.12)

\[
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{[ac_t^{1-b} + (1 - a) m_t^{1-b}]^{b/(1-b)} (1 - a) m_t^{-b}}{[ac_t^{1-b} + (1 - a) m_t^{1-b}]^{b/(1-b)} m_t^{-b}} = \frac{1 - a}{a} \left( \frac{c_t}{m_t} \right)^b = \frac{i_t}{1 + i_t}
\]

- This gives a money demand function as

\[
m_t = \left( \frac{1 - a}{a} \right)^{-\frac{1}{b}} \left( \frac{i_t}{1 + i_t} \right)^{-\frac{1}{b}} c_t \quad (2.31)
\]

- In logs (often used for estimation purposes):

\[
\log m_t = \text{const.} + \log c_t - \frac{1}{b} \log \left( \frac{i_t}{1 + i_t} \right),
\]

where \( 1/b \) is the elasticity of money demand wrt. opportunity cost. I.e., the interest rate elasticity of money demand (depending on money concept, empirically around 0.1–0.3)
Will a unique steady-state value for $m$ exist? Must solve

$$u_m(c^{ss}, m^{ss}) = \left( \frac{i^{ss}}{1 + i^{ss}} \right) u_c(c^{ss}, m^{ss})$$

$$u_m(c^{ss}, m^{ss}) = I^{ss} u_c(c^{ss}, m^{ss})$$

Not necessarily unique.....

Stability properties? For separable utility, $u(c_t, m_t) = v(c_t) + \gamma \phi(m_t)$, resulting difference equation (from first-order condition) will imply a saddle-point stable $m^{ss} > 0$ ($m'$ in Figure 2.1, page 45)

- Problems:
  - One cannot necessarily rule out the paths with increasing $m$ above steady state ("speculative" hyperdeflations)
  - One cannot necessarily rule out the paths with falling $m$ below steady state ("speculative" hyperinflations), leading to $m^{ss} = 0$

In the context of this model, we will not pursue the stability issue
Summary

• MIU model has one for one relationship between inflation and money growth

• MIU model exhibits superneutrality

• A model like the Tobin model is not superneutral; reason is the postulated and policy-invariant private sector behavior. This difference highlights the importance of micro foundations to avoid Lucas critique

• MIU is a structural model where the reactions of the private sector to changes in policy (money growth) are taken into account
  – Potentially more appropriate for analyzing policy changes (even though the micro foundation for money demand is a short cut)
  – (At least for steady-state analyses.)
Plan for next lectures

Tuesday, March 8

1. Money in the utility function (continued)
   a. Welfare costs of inflation
   b. Potential non-superneutrality of money
   c. Dynamics and calibration

   Literature: Walsh (2010, Chapter 2, pp. 52–86, so check the Appendix as well; i.e., get a grip on the linearization technique)

Friday, March 11

Exercises, TBA