Additional Appendix: NOT for publication; for referees use and available upon request

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Abstract

This document provides the Additional Appendix to the paper "Monetary and Fiscal Policy Interactions in a Micro-founded Model of a Monetary Union".

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A Derivation of $V_G\left(G_t^H\right) = v_y\left(Y_t^H, z_t^H\right)$

Setting the derivative of equation (20) with respect to G_t^H to zero, we have:

$$
\begin{aligned}
& \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left[n U_C\left(C_s^H, \epsilon_s^H\right) \frac{\partial C_s^H}{\partial G_t^H}+(1-n) U_C\left(C_s^F, \epsilon_s^F\right) \frac{\partial C_s^F}{\partial G_t^H}\right]+n V_G\left(G_t^H\right) \\
& -\mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left\{n v_y\left(Y_s^H, z_s^H\right)\left[(1-n) T_s^{-n} C_s^W \frac{\partial T_s}{\partial G_t^H}+T_s^{1-n} \frac{\partial C_s^W}{\partial G_t^H}\right]\right\}-n v_y\left(Y_t^H, z_t^H\right) \\
& -\mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left\{(1-n) v_y\left(Y_s^F, z_s^F\right)\left[(-n) T_s^{-(n+1)} C_s^W \frac{\partial T_s}{\partial G_t^H}+T_s^{-n} \frac{\partial C_s^W}{\partial G_t^H}\right]\right\} \\
= & \; 0 .
\end{aligned}
$$

Using (15), (16), (8) and (17) we have that $T_s v_y\left(Y_s^H, z_s^H\right) = v_y\left(Y_s^F, z_s^F\right)$, for all $s \geq t$. Hence, the above expression becomes:

$$
\begin{aligned}
& \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left[n U_C\left(C_s^H, \epsilon_s^H\right) \frac{\partial C_s^H}{\partial G_t^H}+(1-n) U_C\left(C_s^F, \epsilon_s^F\right) \frac{\partial C_s^F}{\partial G_t^H}\right]- \\
& \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} v_y\left(Y_s^H, z_s^H\right)\left\{\begin{array}{c}
n\left[(1-n) T_s^{-n} C_s^W \frac{\partial T_s}{\partial G_t^H}+T_s^{1-n} \frac{\partial C_s^W}{\partial G_t^H}\right]+ \\
(1-n)\left[(-n) T_s^{-n} C_s^W \frac{\partial T_s}{\partial G_t^H}+T_s^{1-n} \frac{\partial C_s^W}{\partial G_t^H}\right]
\end{array}\right\} \\
& +n V_G\left(G_t^H\right)-n v_y\left(Y_t^H, z_t^H\right) \\
= & \; \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left\{\begin{array}{c}
n U_C\left(C_s^H, \epsilon_s^H\right) \frac{\partial C_s^H}{\partial G_t^H}+(1-n) U_C\left(C_s^F, \epsilon_s^F\right) \frac{\partial C_s^F}{\partial G_t^H} \\
-v_y\left(Y_s^H, z_s^H\right) T_s^{1-n} \frac{\partial C_s^W}{\partial G_t^H}
\end{array}\right\} \\
& +n V_G\left(G_t^H\right)-n v_y\left(Y_t^H, z_t^H\right),
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left\{\begin{array}{c}
n U_C\left(C_s^H, \epsilon_s^H\right) \frac{\partial C_s^H}{\partial G_t^H}+(1-n) U_C\left(C_s^F, \epsilon_s^F\right) \frac{\partial C_s^F}{\partial G_t^H} \\
-v_y\left(Y_s^H, z_s^H\right) T_s^{1-n}\left[n \frac{\partial C_s^H}{\partial G_t^H}+(1-n) \frac{\partial C_s^F}{\partial G_t^H}\right]
\end{array}\right\} \\
& +n V_G\left(G_t^H\right)-n v_y\left(Y_t^H, z_t^H\right) \\
= & \; \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left\{n\left[U_C\left(C_s^H, \epsilon_s^H\right)-T_s^{1-n} v_y\left(Y_s^H, z_s^H\right)\right] \frac{\partial C_s^H}{\partial G_t^H}\right\}+ \\
& \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t}\left\{(1-n)\left[U_C\left(C_s^F, \epsilon_s^F\right)-T_s^{1-n} v_y\left(Y_s^H, z_s^H\right)\right] \frac{\partial C_s^F}{\partial G_t^H}\right\}+ \\
& n V_G\left(G_t^H\right)-n v_y\left(Y_t^H, z_t^H\right),
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left\{ n \left[U_C\left(C_s^H, \epsilon_s^H\right) - T_s^{1-n} v_y\left(Y_s^H, z_s^H\right)\right] \frac{\partial C_s^H}{\partial G_t^H} \right\} + \\
& \mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left\{ (1-n) \left[U_C\left(C_s^F, \epsilon_s^F\right) - T_s^{-n} v_y\left(Y_s^F, z_s^F\right)\right] \frac{\partial C_s^F}{\partial G_t^H} \right\} + \\
& n V_G\left(G_t^H\right) - n v_y\left(Y_t^H, z_t^H\right) \\
= \; & n V_G\left(G_t^H\right) - n v_y\left(Y_t^H, z_t^H\right),
\end{aligned}
$$

where in the final step we have used again (15) and (16). Hence,

$$
V_G\left(G_t^H\right) = v_y\left(Y_t^H, z_t^H\right).
$$

Similarly, the first-order condition for Foreign public spending reduces to:

$$
V_G\left(G_t^F\right) = v_y\left(Y_t^F, z_t^F\right).
$$

B Government spending not in utility

B.1 Government spending neither in Home nor in Foreign utility

Government spending does not provide utility to the Home and Foreign individuals. Hence, the coordinating fiscal authorities now maximize

$$
\mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left\{ \begin{array}{c} n \left[U\left(C_s^H, \epsilon_s^H\right) - v\left(Y_s^H, z_s^H\right)\right] \\ +(1-n)\left[U\left(C_s^F, \epsilon_s^F\right) - v\left(Y_s^F, z_s^F\right)\right] \end{array} \right\},
$$

over G_t^H and G_t^F subject to (15), (16) and (17) and subject to the restriction that the steady state values $\overline{G}^H$ and $\overline{G}^F$ of, respectively, G_t^H and G_t^F fulfill $\overline{G}^H = \overline{G}_0^H > 0$ and $\overline{G}^F = \overline{G}_0^F > 0$. For these given steady state values for public spending the best that the authority now can do to maximize welfare is to completely eliminate any fluctuations in the marginal disutility of effort as a result of shocks. Hence, for all t,

$$
\begin{aligned}
v_y\left(Y_t^H, z_t^H\right) &= v_y\left(\overline{Y}_0^H, 0\right), \\
v_y\left(Y_t^F, z_t^F\right) &= v_y\left(\overline{Y}_0^F, 0\right),
\end{aligned}
$$

where $\overline{Y}_0^H = \overline{T}_0^{1-n} \overline{C}_0 + \overline{G}_0^H$ and $\overline{Y}_0^F = \overline{T}_0^{-n} \overline{C}_0 + \overline{G}_0^F$, where $\overline{C}_0$ and $\overline{T}_0$ are the steady state outcomes for consumption and terms of trade when the restrictions are imposed that $\overline{G}^H = \overline{G}_0^H$ and $\overline{G}^F = \overline{G}_0^F$. Log-linearizing the above conditions around this steady state, we obtain $\widetilde{Y}_t^H = S_t^H$ and $\widetilde{Y}_t^F = S_t^F$ which can be rewritten as, respectively,

$$
\begin{aligned}
\widetilde{G}_t^H &= \frac{1}{1-\xi_c} \left[S_t^H - \xi_c \left((1-n) \widetilde{T}_t + \widetilde{C}_t^W \right) \right], \\
\widetilde{G}_t^F &= \frac{1}{1-\xi_c} \left[S_t^F - \xi_c \left(-n \widetilde{T}_t + \widetilde{C}_t^W \right) \right].
\end{aligned}
$$

Hence,

$$\widetilde{G}_t^R = \frac{1}{1-\xi_c}\left[S_t^R + \xi_c \widetilde{T}_t\right].$$

Substitute this into (48), and solve to give $\widetilde{T}_t = 0$.

To show that all gaps can be closed at all dates when government spending does not feature in the utility functions of the individuals, observe that, if $\pi_t^W = 0$ at all dates (to be confirmed), the monetary authority closes the consumption gap by (committing to) setting $\widehat{R}_t = \widetilde{R}_t$, for all $t \geq 1$. From (30) and (31), with $\widehat{G}_t^H - \widetilde{G}_t^H = -\frac{(1+\eta\xi_c)(1-n)}{\eta(1-\xi_c)}\left(\widehat{T}_t - \widetilde{T}_t\right)$, $\widehat{G}_t^F - \widetilde{G}_t^F = \frac{(1+\eta\xi_c)n}{\eta(1-\xi_c)}\left(\widehat{T}_t - \widetilde{T}_t\right)$ and a closed consumption gap for all $t \geq 1$, national inflation rates, and thus π_t^R and π_t^W, are zero for all $t \geq 1$. From (32) we see that, with the natural terms of trade equal to zero for all $t \geq 1$, the starting condition $\widehat{T}_0 = 0$, and $\pi_t^R = 0$ for all $t \geq 1$, the terms-of-trade gap is zero for all $t \geq 1$. Hence, in equilibrium, all gaps are closed and national inflation rates are zero for all $t \geq 1$, and, hence, there are no policy trade offs.

B.2 Government spending in only one country's utility

When only the Home – say – individuals experience utility from public spending, the coordinating fiscal authorities maximize

$$\mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left\{ \begin{array}{c} n\left[U\left(C_s^H, \epsilon_s^H\right) + V\left(G_s^H\right) - v\left(Y_s^H, z_s^H\right)\right] \\ +(1-n)\left[U\left(C_s^F, \epsilon_s^F\right) - v\left(Y_s^F, z_s^F\right)\right] \end{array} \right\},$$

over G_t^H and G_t^F subject to (15), (16) and (17) and subject to the restriction that the steady state $\overline{G}^F$ of G_t^F fulfills $\overline{G}^F = \overline{G}_0^F > 0$. Going through the steps of Additional Appendix A, and using the above arguments, we end up with the following conditions:

$$\begin{aligned} V_G\left(G_t^H\right) &= v_y\left(Y_t^H, z_t^H\right), \\ v_y\left(Y_t^F, z_t^F\right) &= v_y\left(\overline{Y}_0^F, 0\right). \end{aligned}$$

Log-linearizing these two conditions yields:

$$\begin{aligned} \widetilde{G}_t^H &= \frac{\eta}{\rho_g + \eta(1-\xi_c)}\left[S_t^H - \xi_c\left((1-n)\widetilde{T}_t + \widetilde{C}_t^W\right)\right], \\ \widetilde{G}_t^F &= \frac{1}{1-\xi_c}\left[S_t^F - \xi_c\left(-n\widetilde{T}_t + \widetilde{C}_t^W\right)\right]. \end{aligned}$$

These need to be combined with (47) and (48), to yield a four-equation system that is solved for $\widetilde{G}_t^H$, $\widetilde{G}_t^F$, $\widetilde{C}_t^W$ and $\widetilde{T}_t$. We now show that the solution of this system is incompatible with $\widetilde{T}_t = 0$ for all possible shock realizations. By (48), to produce $\widetilde{T}_t = 0$, requires that $\widetilde{G}_t^R = S_t^R/(1-\xi_c)$. From the above expressions for $\widetilde{G}_t^H$ and $\widetilde{G}_t^F$, we obtain a second expression (using $\widetilde{T}_t = 0$):

$$\widetilde{G}_t^R = \frac{1}{1-\xi_c}S_t^F - \frac{\eta}{\rho_g + \eta(1-\xi_c)}S_t^H + \xi_c\left[\frac{\eta}{\rho_g + \eta(1-\xi_c)} - \frac{1}{1-\xi_c}\right]\widetilde{C}_t^W.$$

Choosing $S_t^F = \dfrac{\eta(1-\xi_c)}{\rho_g + \eta(1-\xi_c)} S_t^H$ and working out the term between square brackets, we obtain:

$$\widetilde{G}_t^R = -\frac{\xi_c \rho_g}{\left[\rho_g + \eta(1-\xi_c)\right](1-\xi_c)} \widetilde{C}_t^W.$$

Combining this with

$$\widetilde{G}_t^R = \frac{1}{1-\xi_c} S_t^R = -\frac{\rho_g}{\left[\rho_g + \eta(1-\xi_c)\right](1-\xi_c)} S_t^H,$$

we obtain

$$\widetilde{C}_t^W = S_t^H / \xi_c.$$

From the above expressions for $\widetilde{G}_t^H$ and $\widetilde{G}_t^F$, we also obtain (using $\widetilde{T}_t = 0$):

$$\begin{aligned}
\widetilde{G}_t^W &= \frac{\eta n}{\rho_g + \eta(1-\xi_c)} S_t^H + \frac{1-n}{1-\xi_c} S_t^F - \xi_c \left[\frac{\eta n}{\rho_g + \eta(1-\xi_c)} + \frac{1-n}{1-\xi_c}\right] \widetilde{C}_t^W \\
&= \frac{\eta n}{\rho_g + \eta(1-\xi_c)} S_t^H + \frac{\eta(1-n)}{\rho_g + \eta(1-\xi_c)} S_t^H - \frac{(1-n)\rho_g + \eta(1-\xi_c)}{\left[\rho_g + \eta(1-\xi_c)\right](1-\xi_c)} S_t^H \\
&= \frac{\eta}{\rho_g + \eta(1-\xi_c)} S_t^H - \frac{(1-n)\rho_g + \eta(1-\xi_c)}{\left[\rho_g + \eta(1-\xi_c)\right](1-\xi_c)} S_t^H \\
&= -\frac{(1-n)\rho_g}{\left[\rho_g + \eta(1-\xi_c)\right](1-\xi_c)} S_t^H.
\end{aligned}$$

Substitute this, along with the expression

$$S_t^W = n S_t^H + (1-n)\frac{\eta(1-\xi_c)}{\rho_g + \eta(1-\xi_c)} S_t^H = \frac{n\rho_g + \eta(1-\xi_c)}{\rho_g + \eta(1-\xi_c)} S_t^H,$$

into (47), where, taking the case of $D_t^W = 0$, we get:

$$\begin{aligned}
\widetilde{C}_t^W &= \frac{\eta}{\rho + \eta\xi_c}\left[\frac{n\rho_g + \eta(1-\xi_c)}{\rho_g + \eta(1-\xi_c)} S_t^H + \frac{(1-n)\rho_g}{\left[\rho_g + \eta(1-\xi_c)\right]} S_t^H\right] \\
&= \frac{\eta}{\rho + \eta\xi_c} S_t^H,
\end{aligned}$$

which differs from the earlier expression for $\widetilde{C}_t^W$.

Hence, given that $\widetilde{T}_t$ can differ from zero, we are in the case covered by Proposition 1 and it is impossible to close all gaps.

C Relevance of demand shocks for natural terms of trade under international market incompleteness

As before, log-linearizing (15) and (16) around the steady state, we obtain again (45) and (46). However, subtracting the latter from the former, we can no longer use the relation

$\widetilde{C}_t^H + D_t^H = \widetilde{C}_t^F + D_t^F$ [which was obtained by linearizing (8) – however, this expression no longer holds under internationally incomplete markets]. Hence, we obtain:

$$\rho\left(\widetilde{C}_t^R + D_t^R\right) = \widetilde{T}_t\left(1+\eta\xi_c\right) - \eta\left(1-\xi_c\right)\widetilde{G}_t^R + \eta S_t^R.$$

Hence,

$$\widetilde{T}_t = \frac{1}{1+\eta\xi_c}\left[\rho\left(\widetilde{C}_t^R + D_t^R\right) + \eta\left(1-\xi_c\right)\widetilde{G}_t^R - \eta S_t^R\right].$$

Under internationally complete markets the term $\widetilde{C}_t^R + D_t^R$ would be zero. With international incompleteness this is no longer the case in general.

D Derivation of the microfounded loss function

Here, we derive the utility-based loss function. The average utility flow of the households belonging to countries H and F respectively is:

$$w_t^H = U\left(C_t;\epsilon_t^H\right) + V\left(G_t^H\right) - \frac{1}{n}\int_0^n v\left(y_t\left(h\right), z_t^H\right)\mathrm{d}h,$$

where ϵ_t^H is the vector of Home preference shocks, and

$$w_t^F = U\left(C_t;\epsilon_t^F\right) + V\left(G_t^F\right) - \frac{1}{1-n}\int_n^1 v\left(y_t\left(f\right), z_t^F\right)\mathrm{d}f.$$

The welfare criterion of the authorities (the common central bank and the coordinating fiscal authorities) is:

$$\mathrm{E}_0\sum_{t=1}^{\infty}\beta^{t-1}\left[nw_t^H + \left(1-n\right)w_t^F\right].$$

We start by making computations for Home. The computations for Foreign are analogous and, therefore, not shown explicitly. After this, we combine the expressions for Home and Foreign to obtain W. We denote the full vector of shocks by ξ.

D.1 Working out the terms $U\left(C_t^H;\epsilon_t^H\right)$, $V\left(G_t^H\right)$ and $v\left(y_t\left(h\right);z_t^H\right)$

D.1.1 The term $U\left(C_t^H;\epsilon_t^H\right)$

Take a second-order expansion of $U\left(C_t^H;\epsilon_t^H\right)$ around the steady-state value $\left(\overline{C};0\right)$:

$$\begin{aligned} U\left(C_t^H;\epsilon_t^H\right) &= U\left(\overline{C};0\right) + U_C\left(C_t^H-\overline{C}\right) + \frac{1}{2}U_{CC}\left(C_t^H-\overline{C}\right)^2 \\ &\quad + U_\epsilon\epsilon_t^H + \tfrac{1}{2}\left(\epsilon_t^H\right)' U_{\epsilon\epsilon}\epsilon_t^H + U_{C\epsilon}\epsilon_t^H\left(C_t^H-\overline{C}\right) + \mathcal{O}\left(\|\xi\|^3\right), \end{aligned}$$

where $\mathcal{O}\left(\|\xi\|^3\right)$ stands for terms of third or higher order (all variables are in equilibrium functions of the shock-vector, which exhibits bounded fluctuations of order $\|\xi\|$). Note that a second-order log-expansion of C_t^H around $\overline{C}$ yields:

$$C_t = \overline{C}\left[1 + \widehat{C}_t^H + \frac{1}{2}\left(\widehat{C}_t^H\right)^2\right] + \mathcal{O}\left(\|\xi\|^3\right),$$

where $\widehat{C}_t^H = \ln\left(C_t/\overline{C}\right)$. Substitute the preceding expression into the above expression for $U\left(C_t^H;\epsilon_t^H\right)$ to give:

$$\begin{aligned} U\left(C_t^H;\epsilon_t^H\right) &= U\left(\overline{C};0\right) + U_C\overline{C}\left[\widehat{C}_t^H + \tfrac{1}{2}\left(\widehat{C}_t^H\right)^2\right] + \tfrac{1}{2}U_{CC}\overline{C}^2\left(\widehat{C}_t^H\right)^2 \\ &\quad + U_\epsilon\epsilon_t^H + \tfrac{1}{2}\left(\epsilon_t^H\right)' U_{\epsilon\epsilon}\epsilon_t^H + U_{C\epsilon}\overline{C}\epsilon_t^H\widehat{C}_t^H + \mathcal{O}\left(\|\xi\|^3\right), \end{aligned}$$

and thus

$$\begin{aligned} U\left(C_t^H;\epsilon_t^H\right) &= U_C\overline{C}\left[\widehat{C}_t^H + \tfrac{1}{2}\left(\widehat{C}_t^H\right)^2\right] + \tfrac{1}{2}U_{CC}\overline{C}^2\left(\widehat{C}_t^H\right)^2 + U_{C\epsilon}\overline{C}\epsilon_t^H\widehat{C}_t^H + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right), \\ U\left(C_t^H;\epsilon_t^H\right) &= U_C\overline{C}\left[\widehat{C}_t^H + \tfrac{1}{2}\left(\widehat{C}_t^H\right)^2 + \tfrac{1}{2}\frac{U_{CC}\overline{C}}{U_C}\left(\widehat{C}_t^H\right)^2 + \frac{U_{C\epsilon}}{U_C}\epsilon_t^H\widehat{C}_t^H\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right), \\ U\left(C_t^H;\epsilon_t^H\right) &= U_C\overline{C}\left[\widehat{C}_t^H + \tfrac{1}{2}(1-\rho)\left(\widehat{C}_t^H\right)^2 + \frac{U_{C\epsilon}}{U_C}\epsilon_t^H\widehat{C}_t^H\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right), \end{aligned}$$

where, as in the main text, $\rho \equiv -U_{CC}\left(\overline{C};0\right)\overline{C}/U_C\left(\overline{C};0\right)$, and where "t.i.p." stands for "terms independent of policy." We can then, following the linearization of the first-order conditions, define D_t through the relationship:

$$U_{C\epsilon}\left(\overline{C},0\right)\epsilon_t^H = \overline{C}U_{CC}\left(\overline{C},0\right)D_t^H.$$

This implies that $U_{C\epsilon}\epsilon_t^H/U_C$ is given by $-\rho D_t^H$, and we therefore finally get

$$U\left(C_t^H;\epsilon_t^H\right) = U_C\overline{C}\left[\widehat{C}_t^H + \tfrac{1}{2}(1-\rho)\left(\widehat{C}_t^H\right)^2 - \rho D_t^H\widehat{C}_t^H\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \qquad \text{(A.1)}$$

D.1.2 The term $V\left(G_t^H\right)$

We approximate in an analogous way $V\left(G_t^H\right)$. This yields:

$$V\left(G_t^H\right) = V_G\overline{G}\left[\widehat{G}_t^H + \frac{1}{2}\left(1-\rho_g\right)\left(\widehat{G}_t^H\right)^2\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right),$$

where, in accordance with the main text, $\rho_g \equiv -V_{GG}\left(\overline{G}\right)\overline{G}/V_G\left(\overline{G}\right)$. Using that $V_G\left(\overline{G}\right) = U_C\left(\overline{C};0\right)$, we can write:

$$V\left(G_t^H\right) = U_C\overline{G}\left[\widehat{G}_t^H + \frac{1}{2}\left(1-\rho_g\right)\left(\widehat{G}_t^H\right)^2\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \qquad \text{(A.2)}$$

D.1.3 The term $v\left(y_t(h);z_t^H\right)$

Similarly, we take a second-order Taylor expansion of $v\left(y_t(h);z_t^H\right)$ around a steady state where $y_t(h) = \overline{Y}$ for each h and at each date t, and where $z_t^H = 0$ at each date t. We obtain:

$$\begin{aligned} v\left(y_t(h);z_t^H\right) &= v\left(\overline{Y};0\right) + v_y\left(y_t(h)-\overline{Y}\right) + v_z z_t^H + \tfrac{1}{2}v_{yy}\left(y_t(h)-\overline{Y}\right)^2 \\ &\quad + v_{yz}z_t^H\left(y_t(h)-\overline{Y}\right) + \tfrac{1}{2}\left(z_t^H\right)' v_{zz}z_t^H + \mathcal{O}\left(\|\xi\|^3\right). \end{aligned}$$

Then note that a second-order logarithmic expansion of $y_t(h)$ gives:

$$y_t(h) = \overline{Y}\left[1 + \widehat{y}_t(h) + \tfrac{1}{2}\widehat{y}_t(h)^2\right] + \mathcal{O}\left(\|\xi\|^3\right).$$

Using this expression, we simplify

$$v\left(y_t(h); z_t^H\right) = v_y y_t(h) + \frac{1}{2} v_{yy}\left(y_t(h) - \overline{Y}\right)^2 + v_{yz} z_t^H y_t(h) + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right)$$

to

$$v\left(y_t(h); z_t^H\right) = v_y\overline{Y}\left[\widehat{y}_t(h) + \frac{1}{2}\widehat{y}_t(h)^2 + \frac{1}{2}\frac{v_{yy}\overline{Y}}{v_y}\widehat{y}_t(h)^2 + \frac{v_{yz}}{v_y} z_t^H \widehat{y}_t(h)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right),$$

or

$$v\left(y_t(h); z_t^H\right) = v_y\overline{Y}\left[\widehat{y}_t(h) + \frac{1+\eta}{2}\widehat{y}_t(h)^2 + \frac{v_{yz}}{v_y} z_t^H \widehat{y}_t(h)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right),$$

where, as in the main text, $\eta \equiv v_{yy}\left(\overline{Y}, 0\right)\overline{Y}/v_y\left(\overline{Y}, 0\right)$. Then, using the definition of S_t^H,

$$v_{yz} z_t^H = -\overline{Y} v_{yy} S_t^H,$$

we finally arrive at

$$v\left(y_t(h); z_t^H\right) = v_y\overline{Y}\left[\widehat{y}_t(h) + \frac{1+\eta}{2}\widehat{y}_t(h)^2 - \eta S_t^H \widehat{y}_t(h)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).$$

We have that $U_C\left(\overline{C}; 0\right) = v_y\left(\overline{Y}; 0\right)$. Hence, using this, we can write:

$$v\left(y_t(h); z_t^H\right) = U_C\overline{Y}\left[\widehat{y}_t(h) + \frac{1+\eta}{2}\widehat{y}_t(h)^2 - \eta S_t^H \widehat{y}_t(h)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).$$

This last expression is integrated over the Home population (of size n), i.e., we find

$$\begin{aligned}
&\frac{1}{n}\int_0^n v\left(y_t(h); z_t^H\right) \mathrm{d}h \\
= \;& U_C\overline{Y}\left(\mathrm{E}_h\widehat{y}_t(h) + \tfrac{1+\eta}{2}\left[\mathrm{Var}_h\widehat{y}_t(h) + \left[\mathrm{E}_h\widehat{y}_t(h)\right]^2\right] - \eta S_t^H \mathrm{E}_h\widehat{y}_t(h)\right) + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

We then take a second-order log-expansion of the aggregator Y_t^H to obtain:

$$\widehat{Y}_t^H = \mathrm{E}_h\widehat{y}_t(h) + \tfrac{1}{2}\tfrac{\sigma-1}{\sigma}\mathrm{Var}_h\widehat{y}_t(h) + \mathcal{O}\left(\|\xi\|^3\right).$$

Insert the implied value for $\mathrm{E}_h\widehat{y}_t(h)$ into the previous expression:

$$\begin{aligned}
&\frac{1}{n}\int_0^n v\left(y_t(h); z_t^H\right) \mathrm{d}h \\
= \;& U_C\overline{Y}\left(\widehat{Y}_t^H - \tfrac{1}{2}\tfrac{\sigma-1}{\sigma}\mathrm{Var}_h\widehat{y}_t(h) + \tfrac{1+\eta}{2}\left[\mathrm{Var}_h\widehat{y}_t(h) + \left(\widehat{Y}_t^H\right)^2\right] - \eta S_t^H \widehat{Y}_t^H\right) + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right) \\
= \;& U_C\overline{Y}\left[\widehat{Y}_t^H + \tfrac{1+\eta}{2}\left(\widehat{Y}_t^H\right)^2 - \tfrac{1}{2}\left[\tfrac{\sigma-1}{\sigma} - 1 - \eta\right]\mathrm{Var}_h\widehat{y}_t(h) - \eta S_t^H \widehat{Y}_t^H\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right) \\
= \;& U_C\overline{Y}\left[\widehat{Y}_t^H + \tfrac{1+\eta}{2}\left(\widehat{Y}_t^H\right)^2 + \tfrac{1}{2}\left[\sigma^{-1} + \eta\right]\mathrm{Var}_h\widehat{y}_t(h) - \eta S_t^H \widehat{Y}_t^H\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

D.2 Combining the expressions

Combining (A.1), (A.2) and the previous expression, the relevant welfare criterion is

$$\begin{aligned} w_t^H &= U_C\overline{C}\left[\widehat{C}_t^H + \frac{1}{2}(1-\rho)\left(\widehat{C}_t^H\right)^2 - \rho D_t^H \widehat{C}_t^H\right] \\ &+ U_C\overline{G}\left[\widehat{G}_t^H + \frac{1}{2}(1-\rho_g)\left(\widehat{G}_t^H\right)^2\right] \\ &- U_C\overline{Y}\left[\widehat{Y}_t^H + \tfrac{1+\eta}{2}\left(\widehat{Y}_t^H\right)^2 + \tfrac{1}{2}\left[\sigma^{-1}+\eta\right]\mathrm{Var}_h \widehat{y}_t(h) - \eta S_t^H \widehat{Y}_t^H\right] \\ &+ \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \end{aligned}$$

Using $\xi_c \equiv \overline{C}/\overline{Y}$ (as in the main text) we then get:

$$\begin{aligned} w_t^H &= U_C\overline{C}\left\{\left[\widehat{C}_t^H + \frac{1}{2}(1-\rho)\left(\widehat{C}_t^H\right)^2 - \rho D_t^H \widehat{C}_t^H\right]\right. \\ &+ \frac{1-\xi_c}{\xi_c}\left[\widehat{G}_t^H + \frac{1}{2}(1-\rho_g)\left(\widehat{G}_t^H\right)^2\right] \\ &\left. - \frac{1}{\xi_c}\left[\widehat{Y}_t^H + \tfrac{1+\eta}{2}\left(\widehat{Y}_t^H\right)^2 + \tfrac{1}{2}\left[\sigma^{-1}+\eta\right]\mathrm{Var}_h \widehat{y}_t(h) - \eta S_t^H \widehat{Y}_t^H\right]\right\} \\ &+ \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \end{aligned}$$

For Foreign, we have:

$$\begin{aligned} w_t^F &= U_C\overline{C}\left\{\left[\widehat{C}_t^F + \frac{1}{2}(1-\rho)\left(\widehat{C}_t^F\right)^2 - \rho D_t^F \widehat{C}_t^F\right]\right. \\ &+ \frac{1-\xi_c}{\xi_c}\left[\widehat{G}_t^F + \frac{1}{2}(1-\rho_g)\left(\widehat{G}_t^F\right)^2\right] \\ &\left. - \frac{1}{\xi_c}\left[\widehat{Y}_t^F + \tfrac{1+\eta}{2}\left(\widehat{Y}_t^F\right)^2 + \tfrac{1}{2}\left[\sigma^{-1}+\eta\right]\mathrm{Var}_f \widehat{y}_t(f) - \eta S_t^F \widehat{Y}_t^F\right]\right\} \\ &+ \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \end{aligned}$$

Now, take a weighted average of w_t^H and w_t^F with weights n and $1-n$, respectively:

$$\begin{aligned} w_t &= U_C\overline{C}\left\{\left[\widehat{C}_t^W + \tfrac{1}{2}(1-\rho)\left(\widehat{C}_t^W\right)^2 - \rho D_t^W \widehat{C}_t^W\right]\right. \\ &+ \frac{1-\xi_c}{\xi_c}\left[\widehat{G}_t^W + \tfrac{1}{2}(1-\rho_g)\left(n\left(\widehat{G}_t^H\right)^2 + (1-n)\left(\widehat{G}_t^F\right)^2\right)\right] \\ &\left. - \frac{1}{\xi_c}\left[\begin{array}{c} \widehat{Y}_t^W + \tfrac{1+\eta}{2}\left(n\left(\widehat{Y}_t^H\right)^2 + (1-n)\left(\widehat{Y}_t^F\right)^2\right) + \\ \tfrac{1}{2}\left[\sigma^{-1}+\eta\right]\left[n\mathrm{Var}_h \widehat{y}_t(h) + (1-n)\mathrm{Var}_f \widehat{y}_t(f)\right] \\ -\eta n S_t^H \widehat{Y}_t^H - \eta(1-n) S_t^F \widehat{Y}_t^F \end{array}\right]\right\} \\ &+ \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \end{aligned} \tag{A.3}$$

Here, we have used that:

$$
\begin{aligned}
& n\left(\widehat{C}_t^H\right)^2+(1-n)\left(\widehat{C}_t^F\right)^2= \\
= \; & n\left(\widehat{C}_t^W+D_t^W-D_t^H\right)^2+(1-n)\left(\widehat{C}_t^W+D_t^W-D_t^F\right)^2 \\
= \; & n\left[\left(\widehat{C}_t^W\right)^2+2\left(D_t^W-D_t^H\right)\widehat{C}_t^W\right]+(1-n)\left[\left(\widehat{C}_t^W\right)^2+2\left(D_t^W-D_t^F\right)\widehat{C}_t^W\right]+\text{t.i.p.} \\
= \; & \left(\widehat{C}_t^W\right)^2+\text{t.i.p.}
\end{aligned}
$$

and

$$
\begin{aligned}
& -n\rho D_t^H\widehat{C}_t^H-(1-n)\rho D_t^F\widehat{C}_t^F \\
= \; & -n\rho D_t^H\left(\widehat{C}_t^W+D_t^W-D_t^H\right)-(1-n)\rho D_t^F\left(\widehat{C}_t^W+D_t^W-D_t^F\right) \\
= \; & -n\rho D_t^H\widehat{C}_t^W-(1-n)\rho D_t^F\widehat{C}_t^W+\text{t.i.p.} \\
= \; & -\rho D_t^W\widehat{C}_t^W+\text{t.i.p.}
\end{aligned}
$$

D.2.1 Expansion of $\widehat{Y}_t$

Before continuing, we expand $\widehat{Y}_t^H$. Define the function $W\left(Y_t^H\right)\equiv\widehat{Y}_t^H=\ln\left(Y_t^H/\bar{Y}\right)$. Taking a second-order approximation:

$$
\begin{aligned}
\widehat{Y}_t^H &= W\left(\bar{Y}\right)+W'\left(\bar{Y}\right)\left(Y_t^H-\bar{Y}\right)+\frac{1}{2}W''\left(\bar{Y}\right)\left(Y_t^H-\bar{Y}\right)^2+\mathcal{O}\left(\|\xi\|^3\right) \\
&= 0+\left(\frac{Y_t^H-\bar{Y}}{\bar{Y}}\right)-\frac{1}{2}\left(\frac{Y_t^H-\bar{Y}}{\bar{Y}}\right)^2+\mathcal{O}\left(\|\xi\|^3\right) \\
&= \frac{T_t^{1-n}C_t^W+G_t^H-\left(\bar{T}^{1-n}\bar{C}+\bar{G}\right)}{\bar{Y}}-\frac{1}{2}\left[\frac{T_t^{1-n}C_t^W+G_t^H-\left(\bar{T}^{1-n}\bar{C}+\bar{G}\right)}{\bar{Y}}\right]^2+\mathcal{O}\left(\|\xi\|^3\right) \\
&= \frac{T_t^{1-n}C_t^W-\bar{T}^{1-n}\bar{C}}{\bar{Y}}+\frac{G_t^H-\bar{G}}{\bar{Y}}-\frac{1}{2}\left[\frac{T_t^{1-n}C_t^W-\bar{T}^{1-n}\bar{C}}{\bar{Y}}+\frac{G_t^H-\bar{G}}{\bar{Y}}\right]^2+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

Now define $Z\left(T_t,C_t^W\right)\equiv T_t^{1-n}C_t^W$. Taking a second-order Taylor expansion of $Z\left(T_t,C_t^W\right)$ around the point $\left(\overline{T},\overline{C}\right)$ gives:

$$
\begin{aligned}
Z\left(T_t,C_t^W\right) &= Z\left(\overline{T},\overline{C}\right)+Z_T\left(T_t-\overline{T}\right)+\tfrac{1}{2}Z_{TT}\left(T_t-\overline{T}\right)^2+Z_C\left(C_t^W-\overline{C}\right) \\
&\quad +\tfrac{1}{2}Z_{CC}\left(C_t^W-\overline{C}\right)^2+Z_{TC}\left(T_t-\bar{T}\right)\left(C_t^W-\overline{C}\right)+\mathcal{O}\left(\|\xi\|^3\right) \\
&= \overline{T}^{1-n}\overline{C}+(1-n)\overline{T}^{-n}\overline{C}\left(T_t-\overline{T}\right)-\tfrac{1}{2}(1-n)n\overline{T}^{-(n+1)}\overline{C}\left(T_t-\overline{T}\right)^2 \\
&\quad +\overline{T}^{1-n}\left(C_t^W-\overline{C}\right)+(1-n)\overline{T}^{-n}\left(T_t-\bar{T}\right)\left(C_t^W-\overline{C}\right)+\mathcal{O}\left(\|\xi\|^3\right) \\
&= \overline{T}^{1-n}\overline{C}+(1-n)\overline{T}^{1-n}\overline{C}\left(\frac{T_t-\overline{T}}{\overline{T}}\right)-\tfrac{1}{2}(1-n)n\overline{T}^{1-n}\overline{C}\left(\frac{T_t-\overline{T}}{\overline{T}}\right)^2 \\
&\quad +\overline{T}^{1-n}\overline{C}\left(\frac{C_t^W-\bar{C}}{\bar{C}}\right)+(1-n)\overline{T}^{1-n}\overline{C}\left(\frac{T_t-\overline{T}}{\overline{T}}\right)\left(\frac{C_t^W-\overline{C}}{\overline{C}}\right)+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{T_t^{1-n}C_t^H-\overline{T}^{1-n}\overline{C}}{\bar{Y}} \\
=\;& (1-n)\,\xi_c\left(\frac{T_t-\overline{T}}{\overline{T}}\right)-\tfrac{1}{2}\,(1-n)\,n\xi_c\left(\frac{T_t-\overline{T}}{\overline{T}}\right)^2 \\
&+\xi_c\left(\frac{C_t^W-\overline{C}}{\overline{C}}\right)+(1-n)\,\xi_c\left(\frac{T_t-\overline{T}}{\overline{T}}\right)\left(\frac{C_t^W-\overline{C}}{\overline{C}}\right)+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

Substitute into this expression:

$$
\begin{aligned}
C_t^W &= \bar{C}\left(1+\widehat{C}_t^W+\frac{1}{2}\left(\widehat{C}_t^W\right)^2\right)+\mathcal{O}\left(\|\xi\|^3\right), \\
T_t &= \bar{T}\left(1+\widehat{T}_t+\frac{1}{2}\widehat{T}_t^2\right)+\mathcal{O}\left(\|\xi\|^3\right),
\end{aligned}
$$

so that the right-hand side becomes:

$$
\begin{aligned}
& (1-n)\,\xi_c\left(\widehat{T}_t+\tfrac{1}{2}\widehat{T}_t^2\right)-\tfrac{1}{2}\,(1-n)\,n\xi_c\left(\widehat{T}_t+\tfrac{1}{2}\widehat{T}_t^2\right)^2 \\
& +\xi_c\left(\widehat{C}_t^W+\tfrac{1}{2}\left(\widehat{C}_t^W\right)^2\right)+(1-n)\,\xi_c\left(\widehat{T}_t+\tfrac{1}{2}\widehat{T}_t^2\right)\left(\widehat{C}_t^W+\tfrac{1}{2}\left(\widehat{C}_t^W\right)^2\right)+\mathcal{O}\left(\|\xi\|^3\right) \\
=\;& (1-n)\,\xi_c\widehat{T}_t+\xi_c\widehat{C}_t^W+\tfrac{1}{2}\,(1-n)\,\xi_c\widehat{T}_t^2 \\
& -\tfrac{1}{2}\,(1-n)\,n\xi_c\widehat{T}_t^2+\tfrac{1}{2}\xi_c\left(\widehat{C}_t^W\right)^2+(1-n)\,\xi_c\widehat{T}_t\widehat{C}_t^W+\mathcal{O}\left(\|\xi\|^3\right) \\
=\;& (1-n)\,\xi_c\widehat{T}_t+\xi_c\widehat{C}_t^W+\tfrac{1}{2}\,(1-n)^2\,\xi_c\widehat{T}_t^2+\tfrac{1}{2}\xi_c\left(\widehat{C}_t^W\right)^2+(1-n)\,\xi_c\widehat{T}_t\widehat{C}_t^W+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

In addition using that

$$
G_t^H=\bar{G}\left(1+\widehat{G}_t^H+\frac{1}{2}\left(\widehat{G}_t^H\right)^2\right)+\mathcal{O}\left(\|\xi\|^3\right),
$$

we can write:

$$
\begin{aligned}
\left(\frac{Y_t^H-\overline{Y}}{\overline{Y}}\right) &= \frac{T_t^{1-n}C_t^W-\bar{T}^{1-n}\overline{C}}{\overline{Y}}+\frac{G_t^H-\overline{G}}{\overline{Y}} \\
&= \xi_c\left[(1-n)\,\widehat{T}_t+\widehat{C}_t^W+\tfrac{1}{2}\,(1-n)^2\,\widehat{T}_t^2+\tfrac{1}{2}\left(\widehat{C}_t^W\right)^2+(1-n)\,\widehat{T}_t\widehat{C}_t^W\right] \\
&\quad+(1-\xi_c)\left[\widehat{G}_t^H+\tfrac{1}{2}\left(\widehat{G}_t^H\right)^2\right]+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

Hence:

$$
\begin{aligned}
&\left(\frac{Y_t^H-\overline{Y}}{\overline{Y}}\right)^2 \\
= \;& \xi_c^2\left[(1-n)\widehat{T}_t+\widehat{C}_t^W+\tfrac{1}{2}(1-n)^2\widehat{T}_t^2+\tfrac{1}{2}\left(\widehat{C}_t^W\right)^2+(1-n)\widehat{T}_t\widehat{C}_t^W\right]^2+ \\
& 2\xi_c(1-\xi_c)\left[(1-n)\widehat{T}_t+\widehat{C}_t^W+\tfrac{1}{2}(1-n)^2\widehat{T}_t^2+\tfrac{1}{2}\left(\widehat{C}_t^W\right)^2+(1-n)\widehat{T}_t\widehat{C}_t^W\right]\left[\widehat{G}_t^H+\frac{1}{2}\left(\widehat{G}_t^H\right)^2\right] \\
& +(1-\xi_c)^2\left[\widehat{G}_t^H+\frac{1}{2}\left(\widehat{G}_t^H\right)^2\right]^2+\mathcal{O}\left(\|\xi\|^3\right) \\
= \;& \xi_c^2\left[(1-n)^2\widehat{T}_t^2+\left(\widehat{C}_t^W\right)^2+2(1-n)\widehat{T}_t\widehat{C}_t^W\right]+(1-\xi_c)^2\left(\widehat{G}_t^H\right)^2+ \\
& 2\xi_c(1-\xi_c)(1-n)\widehat{T}_t\widehat{G}_t^H+2\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^H+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left(\frac{Y_t^H-\overline{Y}}{\overline{Y}}\right)-\frac{1}{2}\left(\frac{Y_t^H-\overline{Y}}{\overline{Y}}\right)^2 \\
= \;& \xi_c\left[(1-n)\widehat{T}_t+\widehat{C}_t^W\right]+(1-\xi_c)\widehat{G}_t^H+\tfrac{1}{2}(1-n)^2\xi_c\widehat{T}_t^2+\tfrac{1}{2}\xi_c\left(\widehat{C}_t^W\right)^2+(1-n)\xi_c\widehat{T}_t\widehat{C}_t^W+ \\
& \tfrac{1}{2}(1-\xi_c)\left(\widehat{G}_t^H\right)^2-\tfrac{1}{2}\xi_c^2(1-n)^2\widehat{T}_t^2-\tfrac{1}{2}\xi_c^2\left(\widehat{C}_t^W\right)^2-(1-n)\xi_c^2\widehat{T}_t\widehat{C}_t^W-\tfrac{1}{2}(1-\xi_c)^2\left(\widehat{G}_t^H\right)^2- \\
& \xi_c(1-\xi_c)(1-n)\widehat{T}_t\widehat{G}_t^H-\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^H+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

or

$$
\begin{aligned}
\widehat{Y}_t^H &= \left(\frac{Y_t^H-\overline{Y}}{\overline{Y}}\right)-\frac{1}{2}\left(\frac{Y_t^H-\overline{Y}}{\overline{Y}}\right)^2+\mathcal{O}\left(\|\xi\|^3\right) \\
&= \left[\xi_c\left((1-n)\widehat{T}_t+\widehat{C}_t^W\right)+(1-\xi_c)\widehat{G}_t^H\right] \\
&\quad +\tfrac{1}{2}(1-n)^2\xi_c(1-\xi_c)\widehat{T}_t^2+\tfrac{1}{2}\xi_c(1-\xi_c)\left(\widehat{C}_t^W\right)^2+\tfrac{1}{2}\xi_c(1-\xi_c)\left(\widehat{G}_t^H\right)^2 \\
&\quad +(1-n)\xi_c(1-\xi_c)\widehat{T}_t\widehat{C}_t^W-\xi_c(1-\xi_c)(1-n)\widehat{T}_t\widehat{G}_t^H \\
&\quad -\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^H+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

In a similar way we derive the corresponding expression for the foreign country:

$$
\begin{aligned}
\widehat{Y}_t^F &= \left(\frac{Y_t^F-\overline{Y}}{\overline{Y}}\right)-\frac{1}{2}\left(\frac{Y_t^F-\overline{Y}}{\overline{Y}}\right)^2+\mathcal{O}\left(\|\xi\|^3\right) \\
&= \left[\xi_c\left(-n\widehat{T}_t+\widehat{C}_t^W\right)+(1-\xi_c)\widehat{G}_t^F\right] \\
&\quad +\tfrac{1}{2}n^2\xi_c(1-\xi_c)\widehat{T}_t^2+\tfrac{1}{2}\xi_c(1-\xi_c)\left(\widehat{C}_t^W\right)^2+\tfrac{1}{2}\xi_c(1-\xi_c)\left(\widehat{G}_t^F\right)^2 \\
&\quad -n\xi_c(1-\xi_c)\widehat{T}_t\widehat{C}_t^W+\xi_c(1-\xi_c)n\widehat{T}_t\widehat{G}_t^F \\
&\quad -\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^F+\mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

D.2.2 Continuation of approximation

To further work out the approximation of the welfare loss function, it is useful to obtain some expressions, before actually making the substitutions. We have, using the above expressions:

$$\begin{aligned}
\widehat{Y}_t^W &= \left[\xi_c \widehat{C}_t^W + (1-\xi_c)\,\widehat{G}_t^W\right] + \tfrac{1}{2} n\,(1-n)\,\xi_c\,(1-\xi_c)\,\widehat{T}_t^2 \\
&+ \tfrac{1}{2}\xi_c\,(1-\xi_c)\left[n\left(\widehat{C}_t^W - \widehat{G}_t^H\right)^2 + (1-n)\left(\widehat{C}_t^W - \widehat{G}_t^F\right)^2\right] \\
&+ \xi_c\,(1-\xi_c)\,n\,(1-n)\,\widehat{T}_t \widehat{G}_t^R + \mathcal{O}\left(\|\xi\|^3\right),
\end{aligned}$$

Further,

$$\begin{aligned}
\left(\widehat{Y}_t^H\right)^2 &= \left[\xi_c\left((1-n)\,\widehat{T}_t + \widehat{C}_t^W\right) + (1-\xi_c)\,\widehat{G}_t^H\right]^2 + \mathcal{O}\left(\|\xi\|^3\right), \\
\left(\widehat{Y}_t^F\right)^2 &= \left[\xi_c\left(-n\widehat{T}_t + \widehat{C}_t^W\right) + (1-\xi_c)\,\widehat{G}_t^F\right]^2 + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& n\left(\widehat{Y}_t^H\right)^2 + (1-n)\left(\widehat{Y}_t^F\right)^2 \\
=\; & n\xi_c^2\left[(1-n)\,\widehat{T}_t + \widehat{C}_t^W\right]^2 + (1-n)\,\xi_c^2\left[-n\widehat{T}_t + \widehat{C}_t^W\right]^2 \\
& + 2n\xi_c\,(1-\xi_c)\left[(1-n)\,\widehat{T}_t + \widehat{C}_t^W\right]\widehat{G}_t^H + 2\,(1-n)\,\xi_c\,(1-\xi_c)\left[-n\widehat{T}_t + \widehat{C}_t^W\right]\widehat{G}_t^F \\
& + n\,(1-\xi_c)^2\left(\widehat{G}_t^H\right)^2 + (1-n)\,(1-\xi_c)^2\left(\widehat{G}_t^H\right)^2 + \mathcal{O}\left(\|\xi\|^3\right) \\
=\; & n\xi_c^2\left[(1-n)^2\,\widehat{T}_t^2 + 2\,(1-n)\,\widehat{T}_t\widehat{C}_t^W + \left(\widehat{C}_t^W\right)^2\right] + (1-n)\,\xi_c^2\left[n^2\widehat{T}_t^2 - 2n\widehat{T}_t\widehat{C}_t^W + \left(\widehat{C}_t^W\right)^2\right] \\
& + 2\xi_c\,(1-\xi_c)\left[\widehat{C}_t^W\widehat{G}_t^W - n\,(1-n)\,\widehat{T}_t\widehat{G}_t^R\right] \\
& + n\,(1-\xi_c)^2\left(\widehat{G}_t^H\right)^2 + (1-n)\,(1-\xi_c)^2\left(\widehat{G}_t^H\right)^2 + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& n\left(\widehat{Y}_t^H\right)^2 + (1-n)\left(\widehat{Y}_t^F\right)^2 \\
=\; & n\,(1-n)\,\xi_c^2\widehat{T}_t^2 + \xi_c^2\left(\widehat{C}_t^W\right)^2 + 2\xi_c\,(1-\xi_c)\left[\widehat{C}_t^W\widehat{G}_t^W - n\,(1-n)\,\widehat{T}_t\widehat{G}_t^R\right] \\
& + (1-\xi_c)^2\left[n\left(\widehat{G}_t^H\right)^2 + (1-n)\left(\widehat{G}_t^F\right)^2\right] + \mathcal{O}\left(\|\xi\|^3\right) \\
=\; & n\,(1-n)\,\xi_c^2\widehat{T}_t^2 - 2\xi_c\,(1-\xi_c)\,n\,(1-n)\,\widehat{T}_t\widehat{G}_t^R + \\
& n\left[\xi_c^2\left(\widehat{C}_t^W\right)^2 + 2\xi_c\,(1-\xi_c)\,\widehat{C}_t^W\widehat{G}_t^H + (1-\xi_c)^2\left(\widehat{G}_t^H\right)^2\right] \\
& (1-n)\left[\xi_c^2\left(\widehat{C}_t^W\right)^2 + 2\xi_c\,(1-\xi_c)\,\widehat{C}_t^W\widehat{G}_t^F + (1-\xi_c)^2\left(\widehat{G}_t^F\right)^2\right] + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Further,

$$
\begin{aligned}
S_t^H \widehat{Y}_t^H &= S_t^H \left[\xi_c \left((1-n)\widehat{T}_t + \widehat{C}_t^W\right) + (1-\xi_c)\widehat{G}_t^H\right] + \mathcal{O}\left(\|\xi\|^3\right), \\
S_t^F \widehat{Y}_t^F &= S_t^F \left[\xi_c \left(-n\widehat{T}_t + \widehat{C}_t^W\right) + (1-\xi_c)\widehat{G}_t^F\right] + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \eta\left[nS_t^H \widehat{Y}_t^H + (1-n) S_t^F \widehat{Y}_t^F\right] \\
= \; & \eta n S_t^H \left[\xi_c \left((1-n)\widehat{T}_t + \widehat{C}_t^W\right) + (1-\xi_c)\widehat{G}_t^H\right] \\
& + \eta (1-n) S_t^F \left[\xi_c \left(-n\widehat{T}_t + \widehat{C}_t^W\right) + (1-\xi_c)\widehat{G}_t^F\right] + \mathcal{O}\left(\|\xi\|^3\right) \\
= \; & -\eta\xi_c (1-n) n\widehat{T}_t S_t^R + \eta\xi_c \widehat{C}_t^W S_t^W \\
& + \eta (1-\xi_c)\left[nS_t^H \widehat{G}_t^H + (1-n) S_t^F \widehat{G}_t^F\right] + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}
$$

We can now start to make substitutions into (A.3). First, substitute the expression for $\widehat{Y}_t^W$ and observe that the linear terms cancel. Thus, we have:

$$
\begin{aligned}
& \frac{w_t}{U_C \overline{C}} \\
= \; & \tfrac{1}{2}(1-\rho)\left(\widehat{C}_t^W\right)^2 - \rho D_t^W \widehat{C}_t^W + \tfrac{1-\xi_c}{2\xi_c}(1-\rho_g)\left[n\left(\widehat{G}_t^H\right)^2 + (1-n)\left(\widehat{G}_t^F\right)^2\right] \\
& - \tfrac{1}{2\xi_c} A - \tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h \widehat{y}_t(h) + (1-n)\mathrm{Var}_f \widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \qquad \text{(A.4)}
\end{aligned}
$$

where

$$
\begin{aligned}
A \equiv \; & n(1-n)\xi_c(1-\xi_c)\widehat{T}_t^2 + 2n(1-n)\xi_c(1-\xi_c)\widehat{T}_t\widehat{G}_t^R \\
& + \xi_c(1-\xi_c)\left[n\left(\widehat{C}_t^W - \widehat{G}_t^H\right)^2 + (1-n)\left(\widehat{C}_t^W - \widehat{G}_t^F\right)^2\right] \\
& + (1+\eta) n(1-n)\xi_c^2\widehat{T}_t^2 - 2(1+\eta)\xi_c(1-\xi_c) n(1-n)\widehat{T}_t\widehat{G}_t^R \\
& + (1+\eta) n\left[\xi_c^2\left(\widehat{C}_t^W\right)^2 + 2\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^H + (1-\xi_c)^2\left(\widehat{G}_t^H\right)^2\right] \\
& + (1+\eta)(1-n)\left[\xi_c^2\left(\widehat{C}_t^W\right)^2 + 2\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^F + (1-\xi_c)^2\left(\widehat{G}_t^F\right)^2\right] \\
& - 2\eta\left[nS_t^H\widehat{Y}_t^H + (1-n)S_t^F\widehat{Y}_t^F\right].
\end{aligned}
$$

Substitute this back into (A.4), to give:

$$\begin{aligned}
\frac{w_t}{U_C\overline{C}} &= \tfrac{1}{2}(1-\rho)\left(\widehat{C}_t^W\right)^2 - \rho D_t^W \widehat{C}_t^W + n\tfrac{1-\xi_c}{2\xi_c}(1-\rho_g)\left(\widehat{G}_t^H\right)^2 - n\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W - \widehat{G}_t^H\right)^2 \\
&\quad + (1-n)\tfrac{1-\xi_c}{2\xi_c}(1-\rho_g)\left(\widehat{G}_t^F\right)^2 - (1-n)\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W - \widehat{G}_t^F\right)^2 \\
&\quad - n\tfrac{1+\eta}{2\xi_c}\left[\xi_c^2\left(\widehat{C}_t^W\right)^2 + (1-\xi_c)^2\left(\widehat{G}_t^H\right)^2 + 2\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^H\right] \\
&\quad - (1-n)\tfrac{1+\eta}{2\xi_c}\left[\xi_c^2\left(\widehat{C}_t^W\right)^2 + 2\xi_c(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^F + (1-\xi_c)^2\left(\widehat{G}_t^F\right)^2\right] \\
&\quad - \tfrac{1}{2}n(1-n)(1-\xi_c)\widehat{T}_t^2 - n(1-n)(1-\xi_c)\widehat{T}_t\widehat{G}_t^R \\
&\quad - \tfrac{1}{2}(1+\eta)n(1-n)\xi_c\widehat{T}_t^2 + (1+\eta)(1-\xi_c)n(1-n)\widehat{T}_t\widehat{G}_t^R \\
&\quad + \tfrac{\eta}{\xi_c}\left[nS_t^H\widehat{Y}_t^H + (1-n)S_t^F\widehat{Y}_t^F\right] \\
&\quad - \tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{w_t}{U_C\overline{C}} &= \tfrac{1}{2}\left[1-\rho-\xi_c(1+\eta)\right]\left(\widehat{C}_t^W\right)^2 + n\tfrac{1-\xi_c}{2\xi_c}\left[1-\rho_g-(1-\xi_c)(1+\eta)\right]\left(\widehat{G}_t^H\right)^2 \\
&\quad + (1-n)\tfrac{1-\xi_c}{2\xi_c}\left[1-\rho_g-(1-\xi_c)(1+\eta)\right]\left(\widehat{G}_t^F\right)^2 \\
&\quad - (1+\eta)(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^W - n\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W - \widehat{G}_t^H\right)^2 - (1-n)\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W - \widehat{G}_t^F\right)^2 \\
&\quad - \tfrac{1}{2}n(1-n)(1+\eta\xi_c)\widehat{T}_t^2 + \eta n(1-n)(1-\xi_c)\widehat{T}_t\widehat{G}_t^R \\
&\quad + \tfrac{\eta}{\xi_c}\left[nS_t^H\widehat{Y}_t^H + (1-n)S_t^F\widehat{Y}_t^F\right] - \rho D_t^W\widehat{C}_t^W \\
&\quad - \tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{w_t}{U_C\overline{C}} &= \tfrac{1}{2}\left[1-\rho-\xi_c(1+\eta)\right]\left(\widehat{C}_t^W\right)^2 + n\tfrac{1-\xi_c}{2\xi_c}\left[1-\rho_g-(1-\xi_c)(1+\eta)\right]\left(\widehat{G}_t^H\right)^2 \\
&\quad + (1-n)\tfrac{1-\xi_c}{2\xi_c}\left[1-\rho_g-(1-\xi_c)(1+\eta)\right]\left(\widehat{G}_t^F\right)^2 \\
&\quad - (1+\eta)(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^W - n\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W - \widehat{G}_t^H\right)^2 - (1-n)\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W - \widehat{G}_t^F\right)^2 \\
&\quad - \tfrac{1}{2}n(1-n)(1+\eta\xi_c)\widehat{T}_t^2 + \eta n(1-n)(1-\xi_c)\widehat{T}_t\widehat{G}_t^R \\
&\quad - \left(\rho D_t^W - \eta S_t^W\right)\widehat{C}_t^W - \eta(1-n)n\widehat{T}_tS_t^R + \tfrac{\eta}{\xi_c}(1-\xi_c)\left[nS_t^H\widehat{G}_t^H + (1-n)S_t^F\widehat{G}_t^F\right] \\
&\quad - \tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{w_t}{U_C\overline{C}} &= -\tfrac{1}{2}(\rho+\eta)\left(\widehat{C}_t^W\right)^2 + \tfrac{1}{2}(1-\xi_c)(1+\eta)\left(\widehat{C}_t^W\right)^2 \\
&\quad -n\tfrac{1-\xi_c}{2\xi_c}\left(\rho_g+\eta\right)\left(\widehat{G}_t^H\right)^2 + n\tfrac{1}{2}(1-\xi_c)(1+\eta)\left(\widehat{G}_t^H\right)^2 \\
&\quad -(1-n)\tfrac{1-\xi_c}{2\xi_c}\left(\rho_g+\eta\right)\left(\widehat{G}_t^F\right)^2 + (1-n)\tfrac{1}{2}(1-\xi_c)(1+\eta)\left(\widehat{G}_t^F\right)^2 \\
&\quad -(1+\eta)(1-\xi_c)\widehat{C}_t^W\widehat{G}_t^W - n\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W-\widehat{G}_t^H\right)^2 - (1-n)\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W-\widehat{G}_t^F\right)^2 \\
&\quad -\tfrac{1}{2}n(1-n)(1+\eta\xi_c)\widehat{T}_t^2 + \eta n(1-n)(1-\xi_c)\widehat{T}_t\widehat{G}_t^R \\
&\quad -\left(\rho D_t^W - \eta S_t^W\right)\widehat{C}_t^W - \eta(1-n)n\widehat{T}_t S_t^R + \tfrac{\eta}{\xi_c}(1-\xi_c)\left[nS_t^H\widehat{G}_t^H + (1-n)S_t^F\widehat{G}_t^F\right] \\
&\quad -\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{w_t}{U_C\overline{C}} &= -\tfrac{1}{2}(\rho+\eta)\left(\widehat{C}_t^W\right)^2 + n\tfrac{1}{2}(1-\xi_c)(1+\eta)\left(\widehat{C}_t^W-\widehat{G}_t^H\right)^2 - n\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W-\widehat{G}_t^H\right)^2 \\
&\quad +(1-n)\tfrac{1}{2}(1-\xi_c)(1+\eta)\left(\widehat{C}_t^W-\widehat{G}_t^F\right)^2 - (1-n)\tfrac{1-\xi_c}{2}\left(\widehat{C}_t^W-\widehat{G}_t^F\right)^2 \\
&\quad -n\tfrac{1-\xi_c}{2\xi_c}\left(\rho_g+\eta\right)\left(\widehat{G}_t^H\right)^2 - (1-n)\tfrac{1-\xi_c}{2\xi_c}\left(\rho_g+\eta\right)\left(\widehat{G}_t^F\right)^2 \\
&\quad -\tfrac{1}{2}n(1-n)(1+\eta\xi_c)\widehat{T}_t^2 + \eta n(1-n)(1-\xi_c)\widehat{T}_t\widehat{G}_t^R \\
&\quad -\left(\rho D_t^W - \eta S_t^W\right)\widehat{C}_t^W - \eta(1-n)n\widehat{T}_t S_t^R + \tfrac{\eta}{\xi_c}(1-\xi_c)\left[nS_t^H\widehat{G}_t^H + (1-n)S_t^F\widehat{G}_t^F\right] \\
&\quad -\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{w_t}{U_C\overline{C}} &= -\tfrac{1}{2}(\rho+\eta)\left(\widehat{C}_t^W\right)^2 - \tfrac{1-\xi_c}{2\xi_c}\left(\rho_g+\eta\right)\left[n\left(\widehat{G}_t^H\right)^2 + (1-n)\left(\widehat{G}_t^F\right)^2\right] \\
&\quad +\tfrac{1}{2}(1-\xi_c)\eta\left[n\left(\widehat{C}_t^W-\widehat{G}_t^H\right)^2 + (1-n)\left(\widehat{C}_t^W-\widehat{G}_t^F\right)^2\right] \\
&\quad -\tfrac{1}{2}n(1-n)(1+\eta\xi_c)\widehat{T}_t^2 + \eta n(1-n)(1-\xi_c)\widehat{T}_t\widehat{G}_t^R \\
&\quad -\left(\rho D_t^W - \eta S_t^W\right)\widehat{C}_t^W - \eta(1-n)n\widehat{T}_t S_t^R + \tfrac{\eta}{\xi_c}(1-\xi_c)\left[nS_t^H\widehat{G}_t^H + (1-n)S_t^F\widehat{G}_t^F\right] \\
&\quad -\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Now, express everything in terms of gaps:

$$
\begin{aligned}
&\frac{w_t}{U_C\overline{C}} \\
=\ & -\tfrac{1}{2}(\rho+\eta)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)^2-\tfrac{1-\xi_c}{2\xi_c}\left(\rho_g+\eta\right)\left[n\left(\widehat{G}_t^H-\widetilde{G}_t^H\right)^2+(1-n)\left(\widehat{G}_t^F-\widetilde{G}_t^F\right)^2\right] \\
&+\tfrac{1}{2}(1-\xi_c)\eta\left[n\left(\widehat{C}_t^W-\widetilde{C}_t^W-\left(\widehat{G}_t^H-\widetilde{G}_t^H\right)\right)^2+(1-n)\left(\widehat{C}_t^W-\widetilde{C}_t^W-\left(\widehat{G}_t^F-\widetilde{G}_t^F\right)\right)^2\right] \\
&-\tfrac{1}{2}n(1-n)(1+\eta\xi_c)\left(\widehat{T}_t-\widetilde{T}_t\right)^2+\eta n(1-n)(1-\xi_c)\left(\widehat{T}_t-\widetilde{T}_t\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) \\
& \\
&-(\rho+\eta)\widehat{C}_t^W\widetilde{C}_t^W-n\tfrac{1-\xi_c}{\xi_c}\left(\rho_g+\eta\right)\widehat{G}_t^H\widetilde{G}_t^H-(1-n)\tfrac{1-\xi_c}{\xi_c}\left(\rho_g+\eta\right)\widehat{G}_t^F\widetilde{G}_t^F \\
&+n(1-\xi_c)\eta\widehat{C}_t^W\widetilde{C}_t^W-n(1-\xi_c)\eta\widehat{C}_t^W\widetilde{G}_t^H-n(1-\xi_c)\eta\widetilde{C}_t^W\widehat{G}_t^H+n(1-\xi_c)\eta\widetilde{G}_t^H\widehat{G}_t^H \\
&+(1-n)(1-\xi_c)\eta\widehat{C}_t^W\widetilde{C}_t^W-(1-n)(1-\xi_c)\eta\widehat{C}_t^W\widetilde{G}_t^F-(1-n)(1-\xi_c)\eta\widetilde{C}_t^W\widehat{G}_t^F \\
&+(1-n)(1-\xi_c)\eta\widetilde{G}_t^F\widehat{G}_t^F \\
&-n(1-n)(1+\eta\xi_c)\widehat{T}_t\widetilde{T}_t+\eta n(1-n)(1-\xi_c)\widetilde{T}_t\widehat{G}_t^R+\eta n(1-n)(1-\xi_c)\widehat{T}_t\widetilde{G}_t^R \\
& \\
&-\left(\rho D_t^W-\eta S_t^W\right)\widehat{C}_t^W-\eta(1-n)n\widehat{T}_tS_t^R+\tfrac{\eta}{\xi_c}(1-\xi_c)\left[nS_t^H\widehat{G}_t^H+(1-n)S_t^F\widehat{G}_t^F\right] \\
&-\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h)+(1-n)\mathrm{Var}_f\widehat{y}_t(f)\right]+\text{t.i.p.}+\mathcal{O}\left(\|\xi\|^3\right),
\end{aligned}
$$

where products of natural levels of variables have been put into the "t.i.p.". Simplify the previous expression:

$$
\begin{aligned}
&\frac{w_t}{U_C\overline{C}} \\
=\ & -\tfrac{1}{2}(\rho+\eta\xi_c)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)^2-n\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g+\eta(1-\xi_c)\right]\left(\widehat{G}_t^H-\widetilde{G}_t^H\right)^2 \\
&-(1-n)\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g+\eta(1-\xi_c)\right]\left(\widehat{G}_t^F-\widetilde{G}_t^F\right)^2-n(1-\xi_c)\eta\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)\left(\widehat{G}_t^H-\widetilde{G}_t^H\right) \\
&-(1-n)(1-\xi_c)\eta\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)\left(\widehat{G}_t^F-\widetilde{G}_t^F\right) \\
&-\tfrac{1}{2}n(1-n)(1+\eta\xi_c)\left(\widehat{T}_t-\widetilde{T}_t\right)^2+\eta n(1-n)(1-\xi_c)\left(\widehat{T}_t-\widetilde{T}_t\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) \\
& \\
&-(\rho+\eta\xi_c)\widehat{C}_t^W\widetilde{C}_t^W-(1-\xi_c)\eta\widehat{C}_t^W\widetilde{G}_t^W-(1-\xi_c)\eta\widetilde{C}_t^W\widehat{G}_t^W \\
&-n\tfrac{1-\xi_c}{\xi_c}\left[\rho_g+\eta(1-\xi_c)\right]\widehat{G}_t^H\widetilde{G}_t^H-(1-n)\tfrac{1-\xi_c}{\xi_c}\left[\rho_g+\eta(1-\xi_c)\right]\widehat{G}_t^F\widetilde{G}_t^F \\
& \\
&-n(1-n)(1+\eta\xi_c)\widehat{T}_t\widetilde{T}_t+\eta n(1-n)(1-\xi_c)\widetilde{T}_t\widehat{G}_t^R+\eta n(1-n)(1-\xi_c)\widehat{T}_t\widetilde{G}_t^R \\
& \\
&-\left(\rho D_t^W-\eta S_t^W\right)\widehat{C}_t^W-\eta(1-n)n\widehat{T}_tS_t^R+\tfrac{\eta}{\xi_c}(1-\xi_c)\left[nS_t^H\widehat{G}_t^H+(1-n)S_t^F\widehat{G}_t^F\right] \\
&-\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h)+(1-n)\mathrm{Var}_f\widehat{y}_t(f)\right]+\text{t.i.p.}+\mathcal{O}\left(\|\xi\|^3\right),
\end{aligned}
$$

We can rewrite this expression further as:

$$
\begin{aligned}
&\frac{w_t}{U_C\overline{C}} \\
= \; & -\tfrac{1}{2}\left(\rho+\eta\xi_c\right)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)^2 - n\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g+\eta\left(1-\xi_c\right)\right]\left(\widehat{G}_t^H-\widetilde{G}_t^H\right)^2 \\
& -\left(1-n\right)\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g+\eta\left(1-\xi_c\right)\right]\left(\widehat{G}_t^F-\widetilde{G}_t^F\right)^2 - n\left(1-\xi_c\right)\eta\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)\left(\widehat{G}_t^H-\widetilde{G}_t^H\right) \\
& -\left(1-n\right)\left(1-\xi_c\right)\eta\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)\left(\widehat{G}_t^F-\widetilde{G}_t^F\right) \\
& -\tfrac{1}{2}n\left(1-n\right)\left(1+\eta\xi_c\right)\left(\widehat{T}_t-\widetilde{T}_t\right)^2 + \eta n\left(1-n\right)\left(1-\xi_c\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) \\
& +A_{CW}\widehat{C}_t^W + A_{GH}\widehat{G}_t^H + A_{GF}\widehat{G}_t^F + A_T\widehat{T}_t \\
& -\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t\left(h\right)+\left(1-n\right)\mathrm{Var}_f\widehat{y}_t\left(f\right)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{CW} &= -\left(\rho+\eta\xi_c\right)\widetilde{C}_t^W - \left(1-\xi_c\right)\eta\widetilde{G}_t^W - \left(\rho D_t^W-\eta S_t^W\right) \\
&= -\rho D_t^W + \eta S_t^W - \left(\rho+\eta\right)\widetilde{C}_t^W + \left(1-\xi_c\right)\eta\left(\widetilde{C}_t^W-\widetilde{G}_t^W\right), \\
A_{GH} &= -n\left(1-\xi_c\right)\eta\widetilde{C}_t^W - n\tfrac{1-\xi_c}{\xi_c}\left[\rho_g+\eta\left(1-\xi_c\right)\right]\widetilde{G}_t^H - \eta n\left(1-n\right)\left(1-\xi_c\right)\widetilde{T}_t + \tfrac{\eta}{\xi_c}\left(1-\xi_c\right)nS_t^H, \\
A_{GF} &= -\left(1-n\right)\left(1-\xi_c\right)\eta\widetilde{C}_t^W - \left(1-n\right)\tfrac{1-\xi_c}{\xi_c}\left[\rho_g+\eta\left(1-\xi_c\right)\right]\widetilde{G}_t^F + \eta n\left(1-n\right)\left(1-\xi_c\right)\widetilde{T}_t \\
&\quad +\tfrac{\eta}{\xi_c}\left(1-\xi_c\right)\left(1-n\right)S_t^F, \\
A_T &= -n\left(1-n\right)\left(1+\eta\xi_c\right)\widetilde{T}_t + \eta n\left(1-n\right)\left(1-\xi_c\right)\widetilde{G}_t^R - \eta\left(1-n\right)nS_t^R.
\end{aligned}
$$

We shall now work out these coefficients. However, before doing so, we make use of

$$
\begin{aligned}
\widetilde{G}_t^W &= \frac{\eta\rho}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}\left(S_t^W+\xi_cD_t^W\right), \\
\widetilde{C}_t^W &= \frac{\eta\rho_g}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}S_t^W - \frac{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}D_t^W,
\end{aligned}
$$

so that

$$
\begin{aligned}
\widetilde{C}_t^W-\widetilde{G}_t^W &= \frac{\eta\rho_g}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}S_t^W - \frac{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}D_t^W \\
&\quad - \frac{\eta\rho}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}\left(S_t^W+\xi_cD_t^W\right) \\
&= \frac{\eta\left(\rho_g-\rho\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}S_t^W - \frac{\rho\left(\rho_g+\eta\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}D_t^W,
\end{aligned}
$$

and

$$
\begin{aligned}
& -\frac{1}{\xi_c}\left(\rho_g+\eta\right)\widetilde{G}_t^W-\eta\left(\widetilde{C}_t^W-\widetilde{G}_t^W\right)+\frac{\eta}{\xi_c}S_t^W \\
= & -\frac{1}{\xi_c}\left(\rho_g+\eta\right)\frac{\eta\rho}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}\left(S_t^W+\xi_c D_t^W\right) \\
& -\eta\left[\frac{\eta\left(\rho_g-\rho\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}S_t^W-\frac{\rho\left(\rho_g+\eta\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}D_t^W\right] \\
& +\frac{\eta}{\xi_c}S_t^W \\
= & -\left[\frac{1}{\xi_c}\frac{\eta\rho\left(\rho_g+\eta\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}+\frac{\eta^2\left(\rho_g-\rho\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}-\frac{\eta}{\xi_c}\right]S_t^W \\
& -\left[\frac{\eta\rho\left(\rho_g+\eta\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}-\frac{\eta\rho\left(\rho_g+\eta\right)}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}\right]D_t^W \\
= & -\frac{1}{\xi_c}\frac{\eta}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}\left[\rho\left(\rho_g+\eta\right)+\xi_c\eta\left(\rho_g-\rho\right)-\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]-\eta\xi_c\rho_g\right]S_t^W \\
= & -\frac{1}{\xi_c}\frac{\eta}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}\left[\rho\left(\rho_g+\eta\right)-\xi_c\eta\rho-\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]\right]S_t^W \\
= & -\frac{1}{\xi_c}\frac{\eta}{\rho\left[\rho_g+\eta\left(1-\xi_c\right)\right]+\eta\xi_c\rho_g}\left[\rho\left(\rho_g+\eta\right)-\rho\left[\rho_g+\eta\right]\right]S_t^W \\
= & \ 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
A_{CW} &= -\rho D_t^W + \eta S_t^W - (\rho+\eta)\widetilde{C}_t^W + (1-\xi_c)\eta\left(\widetilde{C}_t^W - \widetilde{G}_t^W\right) \\
&= -\rho D_t^W + \eta S_t^W \\
&\quad -(\rho+\eta)\left[\frac{\eta\rho_g}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}S_t^W - \frac{\rho\left[\rho_g+\eta(1-\xi_c)\right]}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}D_t^W\right] \\
&\quad +(1-\xi_c)\eta\left[\frac{\eta\left(\rho_g-\rho\right)}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}S_t^W - \frac{\rho\left(\rho_g+\eta\right)}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}D_t^W\right] \\
&= \eta\left[1-\frac{(\rho+\eta)\rho_g}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}+\frac{(1-\xi_c)\eta\left(\rho_g-\rho\right)}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}\right]S_t^W \\
&\quad -\rho\left[1-\frac{\left[\rho_g+\eta(1-\xi_c)\right](\rho+\eta)}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}+\frac{(1-\xi_c)\eta\left(\rho_g+\eta\right)}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}\right]D_t^W \\
&= \frac{\eta}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}\left[\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g-(\rho+\eta)\rho_g+(1-\xi_c)\eta\left(\rho_g-\rho\right)\right] \\
&\quad -\frac{\rho}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}\left[\begin{array}{c}\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g \\ -\left[\rho_g+\eta(1-\xi_c)\right](\rho+\eta)+(1-\xi_c)\eta\left(\rho_g+\eta\right)\end{array}\right]D_t^W \\
&= \frac{\eta}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}\left[\rho\rho_g+\eta\xi_c\rho_g-(\rho+\eta)\rho_g+(1-\xi_c)\eta\rho_g\right]S_t^W \\
&\quad -\frac{\rho}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}\left[\rho\rho_g+\eta\xi_c\rho_g-\rho_g(\rho+\eta)-\eta^2(1-\xi_c)+(1-\xi_c)\eta\left(\rho_g+\eta\right)\right] \\
&= -\frac{\rho}{\rho\left[\rho_g+\eta(1-\xi_c)\right]+\eta\xi_c\rho_g}\left[\eta\xi_c\rho_g-\rho_g\eta+(1-\xi_c)\eta\rho_g\right]D_t^W \\
&= 0.
\end{aligned}
$$

and, noting that $\widetilde{G}_t^H = \widetilde{G}_t^W - (1-n)\widetilde{G}_t^R$, $S_t^H = S_t^W - (1-n)S_t^R$ and $\widetilde{T}_t = -\rho_g\widetilde{G}_t^R$:

$$
\begin{aligned}
\tfrac{1}{n}A_{GH} &= -(1-\xi_c)\eta\widetilde{C}_t^W - \tfrac{1-\xi_c}{\xi_c}\left[\rho_g+\eta(1-\xi_c)\right]\left[\widetilde{G}_t^W-(1-n)\widetilde{G}_t^R\right] \\
&\quad +\eta\rho_g(1-n)(1-\xi_c)\widetilde{G}_t^R + \tfrac{\eta}{\xi_c}(1-\xi_c)\left[S_t^W-(1-n)S_t^R\right] \\
&= -\frac{1-\xi_c}{\xi_c}\left(\rho_g+\eta\right)\widetilde{G}_t^W - (1-\xi_c)\eta\left(\widetilde{C}_t^W-\widetilde{G}_t^W\right) + \frac{\eta(1-\xi_c)}{\xi_c}S_t^W \\
&\quad +\tfrac{1-\xi_c}{\xi_c}\left[\rho_g+\eta(1-\xi_c)\right](1-n)\widetilde{G}_t^R + \eta\rho_g(1-n)(1-\xi_c)\widetilde{G}_t^R - \tfrac{\eta}{\xi_c}(1-\xi_c)(1-n)S_t^R \\
&= 0 + \frac{(1-n)(1-\xi_c)}{\xi_c}\left\{\left[\left(\rho_g+\eta(1-\xi_c)\right)+\eta\rho_g\xi_c\right]\widetilde{G}_t^R - \eta S_t^R\right\} \\
&= 0+0=0,
\end{aligned}
$$

and, noting that $\widetilde{G}_t^F = \widetilde{G}_t^W + n\widetilde{G}_t^R$, $S_t^F = S_t^W + nS_t^R$ and $\widetilde{T}_t = -\rho_g\widetilde{G}_t^R$:

$$
\begin{aligned}
\tfrac{1}{1-n}A_{GF} &= -(1-\xi_c)\eta\widetilde{C}_t^W - \tfrac{1-\xi_c}{\xi_c}\left[\rho_g+\eta(1-\xi_c)\right]\left[\widetilde{G}_t^W + n\widetilde{G}_t^R\right] \\
&\quad -\eta\rho_g n(1-\xi_c)\widetilde{G}_t^R + \tfrac{\eta}{\xi_c}(1-\xi_c)\left(S_t^W + nS_t^R\right) \\
&= 0 - \tfrac{n(1-\xi_c)}{\xi_c}\left[\left[\rho_g+\eta(1-\xi_c)\right]\widetilde{G}_t^R + \eta\rho_g\xi_c\widetilde{G}_t^R - \eta S_t^R\right] \\
&= 0+0=0.
\end{aligned}
$$

and, noting that $\widetilde{T}_t = -\rho_g \widetilde{G}_t^R$:

$$\begin{aligned} A_T &= -n(1-n)(1+\eta\xi_c)\widetilde{T}_t + \eta n(1-n)(1-\xi_c)\widetilde{G}_t^R - \eta(1-n)nS_t^R \\ &= -n(1-n)\left[(1+\eta\xi_c)\widetilde{T}_t - \eta(1-\xi_c)\widetilde{G}_t^R + \eta S_t^R\right] \\ &= n(1-n)\left[(1+\eta\xi_c)\rho_g\widetilde{G}_t^R + \eta(1-\xi_c)\widetilde{G}_t^R - \eta S_t^R\right] \\ &= 0+0. \end{aligned}$$

Concluding, we have that:

$$\begin{aligned} &\tfrac{w_t}{U_C\overline{C}} = -\tfrac{1}{2}(\rho+\eta\xi_c)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)^2 - n\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g + \eta(1-\xi_c)\right]\left(\widehat{G}_t^H - \widetilde{G}_t^H\right)^2 \\ &-(1-n)\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g + \eta(1-\xi_c)\right]\left(\widehat{G}_t^F - \widetilde{G}_t^F\right)^2 - n(1-\xi_c)\eta\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\left(\widehat{G}_t^H - \widetilde{G}_t^H\right) \\ &-(1-n)(1-\xi_c)\eta\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\left(\widehat{G}_t^F - \widetilde{G}_t^F\right) \\ &-\tfrac{1}{2}n(1-n)(1+\eta\xi_c)\left(\widehat{T}_t - \widetilde{T}_t\right)^2 + \eta n(1-n)(1-\xi_c)\left(\widehat{T}_t - \widetilde{T}_t\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) \\ &-\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right), \end{aligned}$$

Hence,

$$\begin{aligned} &\tfrac{w_t}{U_C\overline{C}} = -\tfrac{1}{2}(\rho+\eta\xi_c)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)^2 - n\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g + \eta(1-\xi_c)\right]\left(\widehat{G}_t^H - \widetilde{G}_t^H\right)^2 \\ &-(1-n)\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g + \eta(1-\xi_c)\right]\left(\widehat{G}_t^F - \widetilde{G}_t^F\right)^2 - n(1-\xi_c)\eta\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\left(\widehat{G}_t^H - \widetilde{G}_t^H\right) \\ &-(1-n)(1-\xi_c)\eta\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\left(\widehat{G}_t^F - \widetilde{G}_t^F\right) \\ &-\tfrac{1}{2}n(1-n)(1+\eta\xi_c)\left(\widehat{T}_t - \widetilde{T}_t\right)^2 + \eta n(1-n)(1-\xi_c)\left(\widehat{T}_t - \widetilde{T}_t\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) \\ &-\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\mathrm{Var}_h\widehat{y}_t(h) + (1-n)\mathrm{Var}_f\widehat{y}_t(f)\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right). \end{aligned}$$

The final step is to derive $\mathrm{Var}_h\widehat{y}_t(h)$ and $\mathrm{Var}_f\widehat{y}_t(f)$. We have that

$$\mathrm{var}_h\left[\log y_t(h)\right] = \sigma^2\mathrm{var}_h\left[\log p_t(h)\right].$$

We have

$$\begin{aligned} \mathrm{var}_h\left[\log p_t(h)\right] &= \mathrm{var}_h\left[\log p_t(h) - \bar{p}_{t-1}\right] = \mathrm{E}_h\left[\log p_t(h) - \bar{p}_{t-1}\right]^2 - (\Delta\bar{p}_t)^2 \\ &= \alpha^H\mathrm{E}_h\left[\log p_{t-1}(h) - \bar{p}_{t-1}\right]^2 + \left(1-\alpha^H\right)\left[\log\tilde{p}_t(h) - \bar{p}_{t-1}\right]^2 - (\Delta\bar{p}_t)^2 \\ &= \alpha^H\mathrm{var}_h\left[\log p_{t-1}(h)\right] + \left(1-\alpha^H\right)\left[\log\tilde{p}_t(h) - \bar{p}_{t-1}\right]^2 - (\Delta\bar{p}_t)^2, \end{aligned}$$

where

$$\bar{p}_t \equiv \mathrm{E}_h\left[\log p_t(h)\right].$$

Further,

$$\bar{p}_t - \bar{p}_{t-1} = \left(1-\alpha^H\right)\left[\log\tilde{p}_t(h) - \bar{p}_{t-1}\right].$$

Hence,

$$\mathrm{var}_h\left[\log p_t(h)\right] = \alpha^H\mathrm{var}_h\left[\log p_{t-1}(h)\right] + \tfrac{\alpha^H}{1-\alpha^H}(\Delta\bar{p}_t)^2.$$

Using

$$\bar{p}_t = \log P_{H,t} + \mathcal{O}\left(\|\xi\|^2\right),$$

we have:

$$\text{var}_h\left[\log p_t(h)\right] = \alpha^H \text{var}_h\left[\log p_{t-1}(h)\right] + \tfrac{\alpha^H}{1-\alpha^H}\left(\pi_t^H\right)^2 + \mathcal{O}\left(\|\xi\|^3\right).$$

Hence,

$$\begin{aligned}
\text{var}_h\left[\log p_t(h)\right] &= \left(\alpha^H\right)^{t+1}\text{var}_h\left[\log p_{-1}(h)\right] + \sum_{s=0}^{t}\left(\alpha^H\right)^{t-s}\tfrac{\alpha^H}{1-\alpha^H}\left(\pi_s^H\right)^2 + \mathcal{O}\left(\|\xi\|^3\right) \\
&= \sum_{s=0}^{t}\left(\alpha^H\right)^{t-s}\tfrac{\alpha^H}{1-\alpha^H}\left(\pi_s^H\right)^2 + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$\sum_{t=1}^{\infty}\beta^{t-1}\text{var}_h\left[\log p_t(h)\right] = d^H\sum_{t=1}^{\infty}\beta^{t-1}\left(\pi_t^H\right)^2 + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right),$$

where

$$d^H \equiv \frac{\alpha^H}{\left(1-\alpha^H\beta\right)\left(1-\alpha^H\right)}.$$

Similarly, we derive for Foreign:

$$\sum_{t=1}^{\infty}\beta^{t-1}\text{var}_f\left[\log p_t(f)\right] = d^F\sum_{t=1}^{\infty}\beta^{t-1}\left(\pi_t^F\right)^2 + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right),$$

where

$$d^F \equiv \frac{\alpha^F}{\left(1-\alpha^F\beta\right)\left(1-\alpha^F\right)}.$$

Hence, the second-order welfare approximation is given by:

$$\sum_{t=1}^{\infty}\beta^{t-1}\text{E}_0\left[w_t\right],$$

where

$$\begin{aligned}
&\tfrac{w_t}{U_C\overline{C}} = -\tfrac{1}{2}\left(\rho+\eta\xi_c\right)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)^2 - n\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g + \eta\left(1-\xi_c\right)\right]\left(\widehat{G}_t^H - \widetilde{G}_t^H\right)^2 \\
&-\left(1-n\right)\tfrac{1-\xi_c}{2\xi_c}\left[\rho_g + \eta\left(1-\xi_c\right)\right]\left(\widehat{G}_t^F - \widetilde{G}_t^F\right)^2 - \left(1-\xi_c\right)\eta\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) \\
&-\tfrac{1}{2}n\left(1-n\right)\left(1+\eta\xi_c\right)\left(\widehat{T}_t - \widetilde{T}_t\right)^2 + \eta n\left(1-n\right)\left(1-\xi_c\right)\left(\widehat{T}_t - \widetilde{T}_t\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) \\
&-\tfrac{1}{2\xi_c}\tfrac{1+\eta\sigma}{\sigma}\left[n\sigma^2 d^H\left(\pi_t^H\right)^2 + \left(1-n\right)\sigma^2 d^F\left(\pi_t^F\right)^2\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right).
\end{aligned}$$

Hence,

$$w_t = \tfrac{1}{2}U_C\overline{C}\left(1+\eta\sigma\right)\sigma/\xi_c *$$

$$\left\{\begin{array}{c}
-\frac{\xi_c\left(\rho+\eta\xi_c\right)}{\left(1+\eta\sigma\right)\sigma}\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)^2 - \frac{n\left(1-\xi_c\right)\left[\rho_g+\eta\left(1-\xi_c\right)\right]}{\left(1+\eta\sigma\right)\sigma}\left(\widehat{G}_t^H - \widetilde{G}_t^H\right)^2 \\
-\frac{\left(1-n\right)\left(1-\xi_c\right)\left[\rho_g+\eta\left(1-\xi_c\right)\right]}{\left(1+\eta\sigma\right)\sigma}\left(\widehat{G}_t^F - \widetilde{G}_t^F\right)^2 - \frac{2\xi_c\left(1-\xi_c\right)\eta}{\left(1+\eta\sigma\right)\sigma}\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) \\
-\frac{n\left(1-n\right)\xi_c\left(1+\eta\xi_c\right)}{\left(1+\eta\sigma\right)\sigma}\left(\widehat{T}_t - \widetilde{T}_t\right)^2 + \frac{2\eta n\left(1-n\right)\xi_c\left(1-\xi_c\right)}{\left(1+\eta\sigma\right)\sigma}\left(\widehat{T}_t - \widetilde{T}_t\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) \\
-\left[n d^H\left(\pi_t^H\right)^2 + \left(1-n\right)d^F\left(\pi_t^F\right)^2\right] + \text{t.i.p.} + \mathcal{O}\left(\|\xi\|^3\right)
\end{array}\right\}$$

Observing that $\left[nd^H + (1-n)\, d^F\right] = \left[n/k^H + (1-n)/k^F\right]/(1+\eta\sigma)$, ignoring an irrelevant proportionality factor and the terms independent of policy, the second-order welfare loss approximation is given by:

$$L = \sum_{t=1}^{\infty} \beta^{t-1} \mathrm{E}_0 \left[L_t\right],$$

where

$$\begin{aligned} L_t &= \lambda_C \left(\widehat{C}_t^W - \widetilde{C}_t^W\right)^2 + \lambda_T \left(\widehat{T}_t - \widetilde{T}_t\right)^2 + \lambda_G^H \left(\widehat{G}_t^H - \widetilde{G}_t^H\right)^2 + \lambda_G^F \left(\widehat{G}_t^F - \widetilde{G}_t^F\right)^2 + \lambda_\pi^H \left(\pi_t^H\right)^2 + \\ &\quad \lambda_\pi^F \left(\pi_t^F\right)^2 + \lambda_{CG} \left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) - \lambda_{TG} \left(\widehat{T}_t - \widetilde{T}_t\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right). \end{aligned}$$

and

$$\begin{aligned} \lambda_C &\equiv \xi_c \left(\rho + \eta\xi_c\right), \quad \lambda_T \equiv n(1-n)\,\xi_c \left(1+\eta\xi_c\right), \quad \lambda_G^H \equiv n\left(1-\xi_c\right)\left[\rho_g + \eta\left(1-\xi_c\right)\right], \\ \lambda_G^F &\equiv (1-n)\left(1-\xi_c\right)\left[\rho_g + \eta\left(1-\xi_c\right)\right], \quad \lambda_\pi^H \equiv n\sigma/k^H, \quad \lambda_\pi^F \equiv (1-n)\,\sigma/k^F, \\ \lambda_{CG} &\equiv 2\xi_c\left(1-\xi_c\right)\eta, \quad \lambda_{TG} \equiv 2n(1-n)\,\xi_c\left(1-\xi_c\right)\eta. \end{aligned}$$

E Optimal commitment policies with equal rigidities

For convenience, let us define:

$$\begin{aligned} k_T^H &\equiv k^H\left(1+\eta\xi_c\right), \qquad k_T^F \equiv k^F\left(1+\eta\xi_c\right), \\ k_C^H &\equiv k^H\left(\rho+\eta\xi_c\right), \qquad k_C^F \equiv k^F\left(\rho+\eta\xi_c\right), \\ k_G^H &\equiv k^H\eta\left(1-\xi_c\right), \qquad k_G^F \equiv k^F\eta\left(1-\xi_c\right), \end{aligned}$$

Because rigidities are equal, we further define:

$$\begin{aligned} \alpha &\equiv \alpha^H = \alpha^F, \qquad k \equiv k^H = k^F, \\ k_T &\equiv k_T^H = k_T^F, \qquad k_C \equiv k_C^H = k_C^F, \\ k_G &\equiv k_G^H = k_G^F, \end{aligned}$$

so that we can write (30) and (31) as, respectively:

$$\pi_t^H = \beta \mathrm{E}_t \pi_{t+1}^H + (1-n)\, k_T \left(\widehat{T}_t - \widetilde{T}_t\right) + k_C \left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + k_G \left(\widehat{G}_t^H - \widetilde{G}_t^H\right), \quad \text{(A.5)}$$

$$\pi_t^F = \beta \mathrm{E}_t \pi_{t+1}^F - n k_T \left(\widehat{T}_t - \widetilde{T}_t\right) + k_C \left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + k_G \left(\widehat{G}_t^F - \widetilde{G}_t^F\right). \quad \text{(A.6)}$$

To solve for the optimal policies under commitment we set up the relevant Lagrangian (see, e.g., Woodford, 1999a):

$$\begin{aligned} \mathcal{L} &= \mathrm{E}_1 \sum_{t=1}^{\infty} \beta^{t-1} \left\{ L_t \right. \\ &\quad + 2\phi_{1,t} \left[\pi_t^H - \beta\pi_{t+1}^H - k_T(1-n)\left(\widehat{T}_t - \widetilde{T}_t\right) - k_C\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) - k_G\left(\widehat{G}_t^H - \widetilde{G}_t^H\right)\right] \\ &\quad + 2\phi_{2,t} \left[\pi_t^F - \beta\pi_{t+1}^F + k_T n\left(\widehat{T}_t - \widetilde{T}_t\right) - k_C\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) - k_G\left(\widehat{G}_t^F - \widetilde{G}_t^F\right)\right] \\ &\quad \left. + 2\phi_{3,t} \left[\left(\widehat{T}_t - \widetilde{T}_t\right) - \left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) - \pi_t^F + \pi_t^H + \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] \right\}, \end{aligned}$$

where $2\phi_{1,t}$, $2\phi_{2,t}$, and $2\phi_{3,t}$ are the multipliers on (A.5), (A.6), and (32), respectively, and L_t is given by (34). Optimizing over $\widehat{C}_t^W - \widetilde{C}_t^W$, $\widehat{T}_t - \widetilde{T}_t$, π_t^H, π_t^F, $\widehat{G}_t^H - \widetilde{G}_t^H$, and $\widehat{G}_t^F - \widetilde{G}_t^F$ yields the following six necessary first-order conditions for $t \geq 1$,

$$\begin{aligned}
\lambda_C \left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + \lambda_{CG}\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) - \phi_{1,t}k_C - \phi_{2,t}k_C &= 0,\\
\lambda_T \left(\widehat{T}_t - \widetilde{T}_t\right) - \lambda_{TG}\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) - \phi_{1,t}k_T(1-n) + \phi_{2,t}k_T n + \phi_{3,t} - \beta\phi_{3,t+1} &= 0,\\
\lambda_{\pi^H}\pi_t^H + \phi_{1,t} - \phi_{1,t-1} + \phi_{3,t} &= 0,\\
\lambda_{\pi^F}\pi_t^F + \phi_{2,t} - \phi_{2,t-1} - \phi_{3,t} &= 0,\\
n\lambda_G\left(\widehat{G}_t^H - \widetilde{G}_t^H\right) + n\lambda_{CG}\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + \lambda_{TG}\left(\widehat{T}_t - \widetilde{T}_t\right) - \phi_{1,t}k_G &= 0,\\
(1-n)\lambda_G\left(\widehat{G}_t^F - \widetilde{G}_t^F\right) + (1-n)\lambda_{CG}\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) - \lambda_{TG}\left(\widehat{T}_t - \widetilde{T}_t\right) - \phi_{2,t}k_G &= 0.
\end{aligned}$$

Use the values of the loss function parameters to get

$$(k_C\xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + (k_G\xi_c/\sigma)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) - \left(\phi_{1,t} + \phi_{2,t}\right)k_C = 0,$$

$$\begin{aligned}
&(k_T n(1-n)\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right) - (k_G n(1-n)\xi_c/\sigma)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)\\
&-\phi_{1,t}k_T(1-n) + \phi_{2,t}k_T n + \phi_{3,t} - \beta\phi_{3,t+1} = 0,
\end{aligned}$$

$$n\pi_t^H + \phi_{1,t} - \phi_{1,t-1} + \phi_{3,t} = 0,$$

$$(1-n)\pi_t^F + \phi_{2,t} - \phi_{2,t-1} - \phi_{3,t} = 0,$$

$$\begin{aligned}
&n\left(k_G\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^H - \widetilde{G}_t^H\right) + n(k_G\xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\\
&+ (k_G n(1-n)\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right) - \phi_{1,t}k_G = 0,
\end{aligned}$$

$$\begin{aligned}
&(1-n)\left(k_G\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^F - \widetilde{G}_t^F\right) + (1-n)(k_G\xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\\
&- (k_G n(1-n)\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right) - \phi_{2,t}k_G = 0.
\end{aligned}$$

Hence,

$$(\xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + \left(\frac{k_G}{k_C}\xi_c/\sigma\right)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) - \left(\phi_{1,t} + \phi_{2,t}\right) = 0, \tag{A.7}$$

$$\begin{aligned}
&(k_T n(1-n)\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right) - (k_G n(1-n)\xi_c/\sigma)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)\\
&-\phi_{1,t}k_T(1-n) + \phi_{2,t}k_T n + \phi_{3,t} - \beta\phi_{3,t+1} = 0,
\end{aligned}$$

$$n\pi_t^H + \phi_{1,t} - \phi_{1,t-1} + \phi_{3,t} = 0, \tag{A.8}$$

$$(1-n)\pi_t^F + \phi_{2,t} - \phi_{2,t-1} - \phi_{3,t} = 0, \tag{A.9}$$

$$\begin{aligned}
&n\left(\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^H - \widetilde{G}_t^H\right) + n(\xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right)\\
&+ (n(1-n)\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right) - \phi_{1,t} = 0,
\end{aligned} \tag{A.10}$$

$$(1-n)\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^F-\widetilde{G}_t^F\right)+(1-n)\left(\xi_c/\sigma\right)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)$$
$$-\left(n\left(1-n\right)\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)-\phi_{2,t}=0. \qquad \text{(A.11)}$$

Adding the last two conditions gives

$$\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^W-\widetilde{G}_t^W\right)+(\xi_c/\sigma)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)=\phi_{1,t}+\phi_{2,t}.$$

Combine this with the first equation:

$$(\xi_c/\sigma)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)+\left(\frac{k_G}{k_C}\xi_c/\sigma\right)\left(\widehat{G}_t^W-\widetilde{G}_t^W\right)=\phi_{1,t}+\phi_{2,t},$$

to get

$$\begin{aligned}&\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^W-\widetilde{G}_t^W\right)+(\xi_c/\sigma)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)\\ =\;&(\xi_c/\sigma)\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)+\left(\frac{k_G}{k_C}\xi_c/\sigma\right)\left(\widehat{G}_t^W-\widetilde{G}_t^W\right)\Rightarrow\end{aligned}$$

$$\left(\left[\rho_g/\eta+(1-\xi_c)\right]\right)\left(\widehat{G}_t^W-\widetilde{G}_t^W\right)=\left(\frac{k_G}{k_C}\xi_c\right)\left(\widehat{G}_t^W-\widetilde{G}_t^W\right),$$

from which it follows that

$$\widehat{G}_t^W-\widetilde{G}_t^W=0,$$

unless

$$\begin{aligned}\rho_g/\eta+(1-\xi_c)&=\frac{k_G}{k_C}\xi_c\Leftrightarrow\\ \rho_g/\eta+(1-\xi_c)&=\frac{\eta\left(1-\xi_c\right)}{\left(\rho+\eta\xi_c\right)}\xi_c\Leftrightarrow\\ \rho_g\left(\rho+\eta\xi_c\right)/\eta+(1-\xi_c)\left(\rho+\eta\xi_c\right)&=\eta\left(1-\xi_c\right)\xi_c\Leftrightarrow\\ \rho_g\rho/\eta+\rho_g\xi_c+(1-\xi_c)\left(\rho+\eta\xi_c-\eta\xi_c\right)&=0\Leftrightarrow\\ \rho_g/\eta+\rho_g\xi_c/\rho+(1-\xi_c)&=0,\end{aligned}$$

which is never the case. Hence, the world government spending gap is closed under the optimal plan.

Adding (A.8) and (A.9) yields

$$\pi_t^W+\left(\phi_{1,t}+\phi_{2,t}\right)-\left(\phi_{1,t-1}+\phi_{2,t-1}\right)=0,$$

and, therefore, by (A.7)

$$\pi_t^W=-\left(\xi_c/\sigma\right)\left[\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)-\left(\widehat{C}_{t-1}^W-\widetilde{C}_{t-1}^W\right)\right].$$

We now turn to the characterization of relative variables. Equations (A.10) and (A.11) can be rearranged to (by multiplying the first by $(1-n)$ and multiplying the second by n and then subtracting the first from the second)

$$(1-n)\,n\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)$$
$$-\left(n\left(1-n\right)\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)-n\phi_{2,t}+(1-n)\,\phi_{1,t}=0. \qquad \text{(A.12)}$$

From the "inflation equations" (A.8) and (A.9) we get

$$(1-n)\,n\pi_t^R + n\left(\phi_{2,t}-\phi_{2,t-1}\right) - (1-n)\left(\phi_{1,t}-\phi_{1,t-1}\right) - \phi_{3,t} = 0 \Leftrightarrow$$

$$(1-n)\,n\pi_t^R + n\phi_{2,t} - (1-n)\,\phi_{1,t} - n\phi_{2,t-1} + (1-n)\,\phi_{1,t-1} - \phi_{3,t} = 0.$$

Therefore,

$$\begin{aligned}&(1-n)\,n\pi_t^R + (1-n)\,n\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) - \left(n\,(1-n)\,\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\\&-(1-n)\,n\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_{t-1}^R-\widetilde{G}_{t-1}^R\right) + \left(n\,(1-n)\,\xi_c/\sigma\right)\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right) = \phi_{3,t},\end{aligned}$$

$$\begin{aligned}&(1-n)\,n\pi_t^R + (1-n)\,n\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left[\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)-\left(\widehat{G}_{t-1}^R-\widetilde{G}_{t-1}^R\right)\right]\\&-\left(n\,(1-n)\,\xi_c/\sigma\right)\left[\left(\widehat{T}_t-\widetilde{T}_t\right)-\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)\right] = \phi_{3,t},\end{aligned}$$

$$\begin{aligned}&\pi_t^R + \left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left[\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)-\left(\widehat{G}_{t-1}^R-\widetilde{G}_{t-1}^R\right)\right]\\&-\left(\xi_c/\sigma\right)\left[\left(\widehat{T}_t-\widetilde{T}_t\right)-\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)\right] = \phi_{3,t},\end{aligned}$$

or, by use of (32),

$$\begin{aligned}&\pi_t^R + \left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left[\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)-\left(\widehat{G}_{t-1}^R-\widetilde{G}_{t-1}^R\right)\right]\\&-\left(\xi_c/\sigma\right)\left[\pi_t^R-\left(\widetilde{T}_t-\widetilde{T}_{t-1}\right)\right] = \phi_{3,t},\end{aligned}$$

which becomes

$$\begin{aligned}&\pi_t^R\left(1-\xi_c/\sigma\right) + \left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left[\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)-\left(\widehat{G}_{t-1}^R-\widetilde{G}_{t-1}^R\right)\right]\\&+\left(\xi_c/\sigma\right)\left(\widetilde{T}_t-\widetilde{T}_{t-1}\right) = \phi_{3,t}.\end{aligned}$$

Now examine

$$\begin{aligned}&\left(k_T n\,(1-n)\,\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right) - \left(k_G n\,(1-n)\,\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\\&-\phi_{1,t}k_T\,(1-n) + \phi_{2,t}k_T n + \phi_{3,t} - \beta\phi_{3,t+1} = 0 \Leftrightarrow\end{aligned}$$

$$\begin{aligned}&\left(n\,(1-n)\,\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right) - \left(\frac{k_G}{k_T}n\,(1-n)\,\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\\&-\phi_{1,t}\,(1-n) + \phi_{2,t}n + \frac{\phi_{3,t}-\beta\phi_{3,t+1}}{k_T} = 0.\end{aligned}$$

We find $n\phi_{2,t}-(1-n)\,\phi_{1,t}$ from (A.12)

$$\begin{aligned}&(1-n)\,n\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) - \left(n\,(1-n)\,\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\\=\;& n\phi_{2,t}-(1-n)\,\phi_{1,t},\end{aligned}$$

to get:

$$\begin{aligned}
&\left(n\left(1-n\right)\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)-\left(\frac{k_G}{k_T}n\left(1-n\right)\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\\
&+\left(1-n\right)n\left(\left[\rho_g/\eta+\left(1-\xi_c\right)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\\
&-\left(n\left(1-n\right)\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)+\frac{\phi_{3,t}-\beta\phi_{3,t+1}}{k_T}\\
=&\ 0
\end{aligned}$$

$$\left(1-n\right)n\left[\left(\left[\rho_g/\eta+\left(1-\xi_c\right)\right]/\sigma\right)-\frac{k_G}{k_T}\left(\xi_c/\sigma\right)\right]\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)+\frac{\phi_{3,t}-\beta\phi_{3,t+1}}{k_T}=0.$$

Using that

$$\frac{k_G}{k_T}=\frac{\eta\left(1-\xi_c\right)}{1+\eta\xi_c},$$

we get

$$\frac{\left(1-n\right)n}{\sigma}\left[\rho_g/\eta+\left(1-\xi_c\right)-\frac{\eta\left(1-\xi_c\right)\xi_c}{1+\eta\xi_c}\right]\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)+\frac{\phi_{3,t}-\beta\phi_{3,t+1}}{k_T}=0,$$

and then

$$\phi_{3,t}=\beta\phi_{3,t+1}-\frac{k_T\left(1-n\right)n}{\sigma}\left[\frac{\rho_g/\eta+\xi_c\rho_g+\left(1-\xi_c\right)}{1+\eta\xi_c}\right]\left(\widehat{G}_t^R-\widetilde{G}_t^R\right),$$

Hence,

$$\phi_{3,t}=-\frac{k_T\left(1-n\right)n}{\sigma}\left[\frac{\rho_g/\eta+\xi_c\rho_g+\left(1-\xi_c\right)}{1+\eta\xi_c}\right]\sum_{i=0}^{\infty}\beta^i\left(\widehat{G}_{t+i}^R-\widetilde{G}_{t+i}^R\right).$$

To sum up, we have

$$\pi_t^W=-\left(\xi_c/\sigma\right)\left[\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)-\left(\widehat{C}_{t-1}^W-\widetilde{C}_{t-1}^W\right)\right],\quad \widehat{G}_t^W=\widetilde{G}_t^W,$$

$$\begin{aligned}
&\pi_t^R\left(1-\xi_c/\sigma\right)+\left(\left[\rho_g/\eta+\left(1-\xi_c\right)\right]/\sigma\right)\left[\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)-\left(\widehat{G}_{t-1}^R-\widetilde{G}_{t-1}^R\right)\right]\\
&+\left(\xi_c/\sigma\right)\left(\widetilde{T}_t-\widetilde{T}_{t-1}\right)=\phi_{3,t},
\end{aligned}$$

$$\phi_{3,t}=-\frac{k_T\left(1-n\right)n}{\sigma}\left[\frac{\rho_g/\eta+\xi_c\rho_g+\left(1-\xi_c\right)}{1+\eta\xi_c}\right]\sum_{i=0}^{\infty}\beta^i\left(\widehat{G}_{t+i}^R-\widetilde{G}_{t+i}^R\right),$$

which, somewhat rewritten (using (25)), is the system (39)-(41). Together with the Phillips curves the system determines the six endogenous variables $\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)$, $\left(\widehat{G}_t^H-\widetilde{G}_t^H\right)$, $\left(\widehat{G}_t^F-\widetilde{G}_t^F\right)$, π_t^H, π_t^F and $\phi_{3,t}$.

F Optimal discretionary policies with equal rigidities

Before we analyze this problem, we note that for any generic variable X, the following holds:

$$n\left(X^{H}\right)^{2}+(1-n)\left(X^{F}\right)^{2}=\left(X^{W}\right)^{2}+n(1-n)\left(X^{R}\right)^{2}.$$

Using this, we can rewrite L_t as:

$$L_t = L_t^W + n(1-n)\, L_t^R,$$

where

$$L_t^W = \lambda_C^W\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)^2+\lambda_G^W\left(\widehat{G}_t^W-\widetilde{G}_t^W\right)^2+\left(\pi_t^W\right)^2+2\lambda_{CG}^W\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)\left(\widehat{G}_t^W-\widetilde{G}_t^W\right), \tag{A.13}$$

$$L_t^R = \lambda_T^R\left(\widehat{T}_t-\widetilde{T}_t\right)^2+\left(\pi_t^R\right)^2+\lambda_G^R\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)^2-2\lambda_{TG}^R\left(\widehat{T}_t-\widetilde{T}_t\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right), \tag{A.14}$$

where (having multiplied all the preference weights by the common factor k/σ, which leaves all results unaffected):

$$\begin{aligned}
\lambda_C^W &\equiv \frac{k_C\xi_c}{\sigma}, \quad \lambda_G^W \equiv \frac{k_G\left[\rho_g+\eta\left(1-\xi_c\right)\right]}{\eta\sigma}, \quad \lambda_{CG}^W \equiv \frac{k_G\xi_c}{\sigma},\\
\lambda_T^R &\equiv \frac{k_T\xi_c}{\sigma}, \quad \lambda_G^R \equiv \frac{k_G\left[\rho_g+\eta\left(1-\xi_c\right)\right]}{\eta\sigma}, \quad \lambda_{TG}^R \equiv \frac{k_G\xi_c}{\sigma}.
\end{aligned}$$

We observe that the combination (A.5) and (A.6) can be restated in terms of world and relative variables exclusively:

$$\pi_t^W = \beta \mathrm{E}_t \pi_{t+1}^W + k_C\left(\widehat{C}_t^W-\widetilde{C}_t^W\right)+k_G\left(\widehat{G}_t^W-\widetilde{G}_t^W\right), \tag{A.15}$$

$$\pi_t^R = \beta \mathrm{E}_t \pi_{t+1}^R - k_T\left(\widehat{T}_t-\widetilde{T}_t\right)+k_G\left(\widehat{G}_t^R-\widetilde{G}_t^R\right). \tag{A.16}$$

The problem is to minimize the stream of L_t, subject to the constraints (A.15), (A.16) and (32). Since the nominal interest rate can be adjusted freely at no loss, we do not treat equation (27) as a constraint, but assume that the consumption gap is treated as the monetary policy instrument directly, which together with the world government spending gap and the relative government spending gap forms the full set of policy instruments.

Having realized this, part of the discretionary optimization becomes simple; namely the choice of world consumption and world government spending. Notice that these variables do not affect the relative inflation rate, and nor do they affect the terms of trade directly; cf. (A.16) and (32). Equally important, the variables enter the loss function additively separable from the terms of trade and relative government spending. Hence, the optimal choice of the consumption gap and world government spending gap can be cast as a problem of minimizing the discounted sum of L_t^W, taking as given the path of relative inflation rates and the terms of trade, subject to (A.15). This can be labelled as the "world part" of the problem. One can then independently of this determine the optimal relative spending gap as the one that minimizes the discounted sum of L_t^R, taking as given the path of world government spending, the world inflation rate and the consumption gap, and where the minimization is subject to (A.16) and (32). This can be labelled as the "relative part" of the problem. We now turn to solving these two parts.

F.1 "The world part"

The "world part" of the problem reduces to a sequence of static optimization problems of the form

$$\min_{\left(\widehat{C}_t^W - \widetilde{C}_t^W\right),\left(\widehat{G}_t^W - \widetilde{G}_t^W\right)} L_t^W \qquad \text{s.t. (A.15)}$$

taking as given $\mathrm{E}_t \pi_{t+1}^W$, as the period-t consumption gap or government spending gap have no dynamic implications.

Substitute (A.15) into (A.13). Then, the necessary and sufficient first-order conditions for the world problem are:

$$(k_C \xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + k_C \pi_t^W + (k_G \xi_c/\sigma)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) = 0,$$

$$\left(k_G\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) + k_G \pi_t^W + (k_G \xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) = 0.$$

Reducing these equations slightly, reveals the following:

$$\begin{aligned} (\xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) + \pi_t^W + \left(\frac{k_G \xi_c}{k_C \sigma}\right)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) &= 0, \\ \left(\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^W - \widetilde{G}_t^W\right) + \pi_t^W + (\xi_c/\sigma)\left(\widehat{C}_t^W - \widetilde{C}_t^W\right) &= 0. \end{aligned}$$

Hence, world government spending follows as

$$\widehat{G}_t^W - \widetilde{G}_t^W = 0,$$

hence,[1]

$$\pi_t^W = -\frac{\xi_c}{\sigma}\left(\widehat{C}_t^W - \widetilde{C}_t^W\right).$$

F.2 The "relative part"

The "relative part" of the discretionary optimization problem involves, as mentioned, the choice of relative government spending. For this purpose, it is important to acknowledge that this choice only affects the relative inflation rate and the terms of trade. As these terms enter additively in (A.14) (and relative government spending only enters multiplicatively with the terms of trade), the problem "reduces" to one of minimizing the discounted

[1] The generality of this solution requires that

$$\frac{k_G \xi_c}{k_C \sigma} \neq \left[\rho_g/\eta + (1-\xi_c)\right]/\sigma,$$

which is easy to confirm.

sum of (A.14) subject to (A.16) and (32). This problem does not correspond to a sequence of one-period problems, as the choice of $\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)$ affects π_t^R, and thus $\left(\widehat{T}_t - \widetilde{T}_t\right)$ with direct loss implications through the next period's terms of trade (by the dynamics of (32)].

The period-t problem is therefore solved by dynamic programming with past period's terms-of-trade gap as the state variable. I.e., the problem is characterized by the recursion

$$\begin{aligned} & V\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) \\ = & \min_{\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)} \mathrm{E}_{t-1}\left\{\left(k_T\xi_c/\sigma\right)\left(\widehat{T}_t - \widetilde{T}_t\right)^2 + \left(k_G\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)^2 \right. \\ & \left. + \left(\pi_t^R\right)^2 - 2\left(k_G\xi_c/\sigma\right)\left(\widehat{T}_t - \widetilde{T}_t\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) + \beta V\left(\widehat{T}_t - \widetilde{T}_t\right)\right\}, \end{aligned}$$

where V is the "value" function, and where the minimization is subject to (A.16) and (32). Now, combine these constraints to

$$\begin{aligned} \pi_t^R &= \beta \mathrm{E}_t \pi_{t+1}^R - k_T\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) + \pi_t^R - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] \\ & + k_G\left(\widehat{G}_t^R - \widetilde{G}_t^R\right). \end{aligned}$$

To proceed with the solution, assume that $\widetilde{T}_t - \widetilde{T}_{t-1}$ follows an $AR(1)$ process.[2] Therefore we conjecture that the solution to the relative variables will be linear functions of the state and driving variables. I.e., we conjecture that

$$\pi_t^R = -b_1\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) + b_2\left(\widetilde{T}_t - \widetilde{T}_{t-1}\right), \qquad \text{(A.17)}$$

where b_1 and b_2 are unknown coefficients to be determined. By use of (A.17) one obtains relative inflation as

$$\begin{aligned} \pi_t^R &= -b_1\beta\left(\widehat{T}_t - \widetilde{T}_t\right) + \beta \mathrm{E}_t\left[b_2\left(\widetilde{T}_{t+1} - \widetilde{T}_t\right)\right] \\ & - k_T\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) + \pi_t^R - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] + k_G\left(\widehat{G}_t^R - \widetilde{G}_t^R\right), \end{aligned}$$

$$\begin{aligned} \pi_t^R &= -b_1\beta\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) + \pi_t^R - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] + \beta \mathrm{E}_t\left[b_2\left(\widetilde{T}_{t+1} - \widetilde{T}_t\right)\right] \\ & - k_T\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) + \pi_t^R - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] + k_G\left(\widehat{G}_t^R - \widetilde{G}_t^R\right), \end{aligned}$$

$$\begin{aligned} \pi_t^R\left(1 + b_1\beta + k_T\right) &= -\left(b_1\beta + k_T\right)\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] \\ & + k_G\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) + \beta \mathrm{E}_t\left[b_2\left(\widetilde{T}_{t+1} - \widetilde{T}_t\right)\right], \end{aligned}$$

[2] As $\widetilde{T}_t - \widetilde{T}_{t-1}$ is a linear function of the underlying productivity shocks, we could also have assumed that these shocks followed $AR(1)$ (or more general) processes and formulated the conjecture in terms of the state and all these shocks. This, however, would make the exposition more messy, without affecting the characterization of optimal relative spending gaps; cf. below.

$$\pi_t^R = -\frac{b_1\beta + k_T}{1 + b_1\beta + k_T}\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] + \frac{k_G}{1 + b_1\beta + k_T}\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) + \frac{\beta \mathrm{E}_t\left[b_2\left(\widetilde{T}_{t+1} - \widetilde{T}_t\right)\right]}{1 + b_1\beta + k_T}, \tag{A.18}$$

and the terms-of-trade gap as

$$\begin{aligned}\widehat{T}_t - \widetilde{T}_t = {} & \left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) \\ & -\frac{b_1\beta + k_T}{1 + b_1\beta + k_T}\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] \\ & +\frac{k_G}{1 + b_1\beta + k_T}\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) + \frac{\beta \mathrm{E}_t\left[b_2\left(\widetilde{T}_{t+1} - \widetilde{T}_t\right)\right]}{1 + b_1\beta + k_T} \\ & -\left(\widetilde{T}_t - \widetilde{T}_{t-1}\right),\end{aligned}$$

and, thus,

$$\widehat{T}_t - \widetilde{T}_t = \frac{1}{1 + b_1\beta + k_T}\left[\left(\widehat{T}_{t-1} - \widetilde{T}_{t-1}\right) - \left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)\right] + \frac{k_G}{1 + b_1\beta + k_T}\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) + \frac{\beta \mathrm{E}_t\left[b_2\left(\widetilde{T}_{t+1} - \widetilde{T}_t\right)\right]}{1 + b_1\beta + k_T}. \tag{A.19}$$

One can then insert (A.18) and (A.19) into the value function and obtain an unconstrained minimization problem. The first-order condition for optimal $\widehat{G}_t^R - \widetilde{G}_t^R$ is

$$\begin{aligned}& (k_T\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right)\frac{\partial\left(\widehat{T}_t - \widetilde{T}_t\right)}{\partial\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)} + \left(k_G\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) \\ & + \left(\pi_t^R\right)\frac{\partial \pi_t^R}{\partial\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)} - (k_G\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right) \\ & - (k_G\xi_c/\sigma)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)\frac{\partial\left(\widehat{T}_t - \widetilde{T}_t\right)}{\partial\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)} + \frac{1}{2}\beta V'\left(\widehat{T}_t - \widetilde{T}_t\right)\frac{\partial\left(\widehat{T}_t - \widetilde{T}_t\right)}{\partial\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)} \\ = {} & 0,\end{aligned}$$

or,

$$\begin{aligned}& (k_T\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right)\frac{k_G}{1 + b_1\beta + k_T} + \left(k_G\left[\rho_g/\eta + (1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right) \\ & + \left(\pi_t^R\right)\frac{k_G}{1 + b_1\beta + k_T} - (k_G\xi_c/\sigma)\left(\widehat{T}_t - \widetilde{T}_t\right) - (k_G\xi_c/\sigma)\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)\frac{k_G}{1 + b_1\beta + k_T} \\ & +\frac{1}{2}\beta V'\left(\widehat{T}_t - \widetilde{T}_t\right)\frac{k_G}{1 + b_1\beta + k_T} \\ = {} & 0.\end{aligned} \tag{A.20}$$

Differentiating the value function with respect to $\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)$ yields:

$$\begin{aligned}
\frac{1}{2}V'\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right) &= \left(k_T\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{\partial\left(\widehat{T}_t-\widetilde{T}_t\right)}{\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)} \\
&+ \left(k_G\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\frac{\partial\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)}{\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)} \\
&+ \left(\pi_t^R\right)\frac{\partial\pi_t^R}{\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)} - \left(k_G\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{\partial\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)}{\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)} \\
&- \left(k_G\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\frac{\partial\left(\widehat{T}_t-\widetilde{T}_t\right)}{\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)} \\
&+ \frac{1}{2}\beta V'\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{\partial\left(\widehat{T}_t-\widetilde{T}_t\right)}{\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)}.
\end{aligned}$$

By the Envelope Theorem, we eliminate all terms involving $\partial\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)/\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)$ [the explicit ones and those implicitly appearing in $\partial\left(\widehat{T}_t-\widetilde{T}_t\right)/\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)$ and $\partial\pi_t^R/\partial\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)$] to get:

$$\begin{aligned}
\frac{1}{2}V'\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right) &= \left(k_T\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{1}{1+b_1\beta+k_T} - \pi_t^R\frac{b_1\beta+k_T}{1+b_1\beta+k_T} \\
&- \left(k_G\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\frac{1}{1+b_1\beta+k_T} \\
&+ \frac{1}{2}\beta V'\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{1}{1+b_1\beta+k_T}.
\end{aligned} \tag{A.21}$$

Multiply on both sides by k_G to get

$$\begin{aligned}
\frac{k_G}{2}V'\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right) &= \left(k_T\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{k_G}{1+b_1\beta+k_T} - \pi_t^R k_G\frac{b_1\beta+k_T}{1+b_1\beta+k_T} \\
&- \left(k_G\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\frac{k_G}{1+b_1\beta+k_T} \\
&+ \frac{1}{2}\beta V'\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{k_G}{1+b_1\beta+k_T},
\end{aligned}$$

and add this to (A.20):

$$
\begin{aligned}
& \left(k_T\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{k_G}{1+b_1\beta+k_T}+\left(k_G\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)+ \\
& \left(\pi_t^R\right)\frac{k_G}{1+b_1\beta+k_T}-\left(k_G\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)-\left(k_G\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\frac{k_G}{1+b_1\beta+k_T} \\
& +\frac{1}{2}\beta V'\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{k_G}{1+b_1\beta+k_T}+\frac{k_G}{2}V'\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right) \\
= & \left(k_T\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{k_G}{1+b_1\beta+k_T}-\pi_t^R k_G\frac{b_1\beta+k_T}{1+b_1\beta+k_T} \\
& -\left(k_G\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\frac{k_G}{1+b_1\beta+k_T}+\frac{1}{2}\beta V'\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{k_G}{1+b_1\beta+k_T},
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
& \left(k_G\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)+\left(\pi_t^R\right)\frac{k_G}{1+b_1\beta+k_T} \\
& -\left(k_G\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)+\frac{k_G}{2}V'\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right) \\
= & -\pi_t^R k_G\frac{b_1\beta+k_T}{1+b_1\beta+k_T},
\end{aligned}
$$

from which one gets

$$
\frac{1}{2}V'\left(\widehat{T}_{t-1}-\widetilde{T}_{t-1}\right)=\left(\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)-\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)-\pi_t^R.
$$

Forward this one period, and use it in (A.20) to eliminate the derivative of the value function:

$$
\begin{aligned}
& \left(k_T\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)\frac{k_G}{1+b_1\beta+k_T}+\left(k_G\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) \\
& +\left(\pi_t^R\right)\frac{k_G}{1+b_1\beta+k_T}-\left(k_G\xi_c/\sigma\right)\left(\widehat{T}_t-\widetilde{T}_t\right)-\left(k_G\xi_c/\sigma\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)\frac{k_G}{1+b_1\beta+k_T} \\
& +\frac{\beta k_G}{1+b_1\beta+k_T}\left[\left(\xi_c/\sigma\right)\mathrm{E}_t\left(\widehat{T}_{t+1}-\widetilde{T}_{t+1}\right)-\left(\left[\rho_g/\eta+(1-\xi_c)\right]/\sigma\right)\mathrm{E}_t\left(\widehat{G}_{t+1}^R-\widetilde{G}_{t+1}^R\right)-\mathrm{E}_t\pi_{t+1}^R\right] \\
= & \ 0,
\end{aligned}
$$

and thus

$$
\begin{aligned}
& \frac{\xi_c k_T}{1+b_1\beta+k_T}\left(\widehat{T}_t-\widetilde{T}_t\right)+\left[\rho_g/\eta+(1-\xi_c)\right]\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) \\
& +\frac{\sigma}{1+b_1\beta+k_T}\pi_t^R-\xi_c\left(\widehat{T}_t-\widetilde{T}_t\right)-\frac{\xi_c k_G}{1+b_1\beta+k_T}\left(\widehat{G}_t^R-\widetilde{G}_t^R\right) \\
& +\frac{\beta}{1+b_1\beta+k_T}\left[\xi_c\mathrm{E}_t\left(\widehat{T}_{t+1}-\widetilde{T}_{t+1}\right)-\left[\rho_g/\eta+(1-\xi_c)\right]\mathrm{E}_t\left(\widehat{G}_{t+1}^R-\widetilde{G}_{t+1}^R\right)-\sigma\mathrm{E}_t\pi_{t+1}^R\right] \\
= & \ 0,
\end{aligned}
$$

or,

$$
\begin{aligned}
& -\frac{\xi_c\left(1+b_1\beta\right)}{1+b_1\beta+k_T}\left(\widehat{T}_t-\widetilde{T}_t\right)+\left(\mu-\frac{\xi_c k_G}{1+b_1\beta+k_T}\right)\left(\widehat{G}_t^R-\widetilde{G}_t^R\right)+\frac{\sigma}{1+b_1\beta+k_T}\pi_t^R \\
& +\frac{\beta}{1+b_1\beta+k_T}\left[\xi_c\mathrm{E}_t\left(\widehat{T}_{t+1}-\widetilde{T}_{t+1}\right)-\mu\mathrm{E}_t\left(\widehat{G}_{t+1}^R-\widetilde{G}_{t+1}^R\right)-\sigma\mathrm{E}_t\pi_{t+1}^R\right] \\
= & \ 0,
\end{aligned}
$$

with

$$\mu \equiv \rho_g/\eta + (1 - \xi_c).$$

This is further reduced to

$$\begin{aligned}
& -\xi_c (1 + b_1\beta) \left(\widehat{T}_t - \widetilde{T}_t\right) + (\mu [1 + b_1\beta + k_T] - \xi_c k_G) \left(\widehat{G}_t^R - \widetilde{G}_t^R\right) + \sigma\pi_t^R \\
& +\beta \left[\xi_c \mathrm{E}_t \left(\widehat{T}_{t+1} - \widetilde{T}_{t+1}\right) - \mu \mathrm{E}_t \left(\widehat{G}_{t+1}^R - \widetilde{G}_{t+1}^R\right) - \sigma \mathrm{E}_t \pi_{t+1}^R\right] \\
= \ & 0. \qquad (A.22)
\end{aligned}$$

This equation will, together with (A.18) and (A.19), provide solutions for the paths for $\left(\widehat{G}_t^R - \widetilde{G}_t^R\right)$, $\left(\widehat{T}_t - \widetilde{T}_t\right)$ and π_t^R as functions of the state and $\left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)$. Given the assumption about the stochastic properties of $\left(\widetilde{T}_t - \widetilde{T}_{t-1}\right)$, the solution can be characterized by the method of undertermined coefficients. The coefficients found in this step will be functions of the unknown parameters b_1 and b_2. These are then finally identified by equating the coefficients in the solution for π_t^R with those in the conjecture (A.17).

Although (A.22) is a rather involved expression, the crucial difference with the corresponding relationship (40) under commitment is that $\left(\widehat{G}_{t-1}^R - \widetilde{G}_{t-1}^R\right)$ is *absent* from (A.22). In other words, the history dependence that characterizes the optimal relative government spending policies under commitment is absent under discretionary policymaking.

Note that indeed only the undetermined coefficient to the state variable appears in the characterization of the solution of the system of relative variables as given by equation (A.22). Hence, had we replaced (A.17), by a linear conjecture which depended on the state and the underlying shocks [and assumed that these shocks were $AR(1)$ or more general processes], we would have arrived at the same characterization of optimal relative spending gaps as equation (A.22). The reason is that the impact of government spending changes on the inflation differential and the terms-of-trade-gap only depends on the undetermined coefficient on the state variable. The coefficients on the shocks do therefore not affect the first-order condition or the envelope condition [see equations (A.20) and (A.21)]. We can therefore without loss of analytical generality arrive at equation (A.22) with our parsimonious conjecture (A.17) as claimed in Footnote 2 of this Additional Appendix.