Written Exam at the Department of Economics winter 2018-19

Contract Theory

Final Exam

January 10, 2019

(3-hour closed book exam)

Answers only in English.

This exam consists of five pages in total, including this one

NB: If you fall ill during an examination at Peter Bangs Vej, you must contact an invigilator who will show you how to register and submit a blank exam paper. Then you leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Be careful not to cheat at exams!

- You cheat at an exam, if during the exam, you:
- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Attempt both questions.
Explain all the steps of your analysis and define any new notation that you use.
Show all the calculations that your analysis relies on.

Question 1: Information gathering in the adverse selection model

Consider the following extension of the basic adverse selection model.[1] A firm (the agent) interacts with a government procurement agency (the principal). The firm produces office material that the procurement agency wants to buy. The firm's cost of producing q units of office material is given by the function $C(q,\theta)$, where θ is an efficiency parameter. This function satisfies

$$C(0,\theta) = 0, \qquad C_q > 0, \qquad C_{qq} \geq 0, \qquad C_\theta > 0, \qquad C_{q\theta} > 0, \qquad C_{qq\theta} \geq 0.$$

The value for the procurement agency of receiving q units of office material is given by the function $S(q)$, which satisfies

$$S'(q) > 0, \qquad S''(q) < 0, \qquad S(0) = 0.$$

The efficiency parameter θ can take two values: $\theta \in \{\underline{\theta}, \overline{\theta}\}$, with $0 < \underline{\theta} < \overline{\theta}$. Initially (and this is where the model differs from the standard adverse selection model), neither the firm nor the procurement agency knows the value of θ: they both believe that

$$\Pr\left[\theta = \underline{\theta}\right] = \nu \qquad \text{and} \qquad \Pr\left[\theta = \overline{\theta}\right] = 1 - \nu,$$

with $0 < \nu < 1$. However, the firm can, if incurring a cost $\gamma > 0$, learn the value of θ. The timing of events is as follows.

1. The procurement agency chooses a menu of contracts. A contract can specify the quantity q that the firm must produce and deliver and the payment t that the firm will receive.

2. The firm decides whether or not to incur information gathering costs γ to learn θ. The procurement agency cannot observe whether the firm incurs γ, nor can it observe the value of θ that the firm possibly learns.

3. The firm decides whether to reject all contracts in the menu or to accept one of them.

[1] The problem is identical to one that we studied in a problem set in the course, except that the questions are shortened and simplified somewhat.

4. If the firm accepted a contract at date 3, production takes place and the procurement agency pays the firm the contractually specified payment t.

The procurement agency is risk neutral and its payoff, given a quantity q and a payment t, equals $V = S(q) - t$. The firm is also risk neutral and its payoff, given a quantity q and a payment t, equals $U = t - C(q,\theta) - \gamma$ if it has gathered information at date 2 and $U = t - C(q,\theta)$ otherwise. If the firm rejects all contracts at date 3, then its payoff equals $-\gamma$ if it has gathered information at date 2 and zero otherwise.

Suppose the procurement agency wants to induce the firm to gather information. Also suppose that the parameters of the model are such that it is optimal to interact with both types and to offer them distinct contracts. Then we can write the procurement agency's problem as follows. The principal chooses $\left(\underline{q},\overline{q},\underline{t},\overline{t}\right)$ so as to maximize its expected payoff,

$$V\left(\underline{t},\underline{q},\overline{t},\overline{q}\right) = \nu\left[S\left(\underline{q}\right) - \underline{t}\right] + (1-\nu)\left[S(\overline{q}) - \overline{t}\right],$$

subject to seven constraints:

$$\overline{t} - C\left(\overline{q},\overline{\theta}\right) \geq 0, \qquad \text{(IR-bad)}$$

$$\underline{t} - C\left(\underline{q},\underline{\theta}\right) \geq 0, \qquad \text{(IR-good)}$$

$$\overline{t} - C\left(\overline{q},\overline{\theta}\right) \geq \underline{t} - C\left(\underline{q},\overline{\theta}\right), \qquad \text{(IC-bad)}$$

$$\underline{t} - C\left(\underline{q},\underline{\theta}\right) \geq \overline{t} - C\left(\overline{q},\underline{\theta}\right), \qquad \text{(IC-good)}$$

$$\nu\left[\underline{t} - C\left(\underline{q},\underline{\theta}\right)\right] + (1-\nu)\left[\overline{t} - C\left(\overline{q},\overline{\theta}\right)\right] - \gamma \geq \underline{t} - \nu C\left(\underline{q},\underline{\theta}\right) - (1-\nu)\,C\left(\underline{q},\overline{\theta}\right), \qquad \text{(IG-good)}$$

$$\nu\left[\underline{t} - C\left(\underline{q},\underline{\theta}\right)\right] + (1-\nu)\left[\overline{t} - C\left(\overline{q},\overline{\theta}\right)\right] - \gamma \geq \overline{t} - \nu C\left(\overline{q},\underline{\theta}\right) - (1-\nu)\,C\left(\overline{q},\overline{\theta}\right), \qquad \text{(IG-bad)}$$

$$\nu\left[\underline{t} - C\left(\underline{q},\underline{\theta}\right)\right] + (1-\nu)\left[\overline{t} - C\left(\overline{q},\overline{\theta}\right)\right] - \gamma \geq 0. \qquad \text{(IR-ante)}$$

(a) Explain (briefly) in words what each one of the three last constraints says—i.e., interpret IG-good, IG-bad, and IR-ante.

Let the first-best quantities, $\overline{q}^{FB}$ and $\underline{q}^{FB}$, be defined in the usual way by $S'\left(\overline{q}^{FB}\right) = C_q\left(\overline{q}^{FB},\overline{\theta}\right)$ and $S'\left(\underline{q}^{FB}\right) = C_q\left(\underline{q}^{FB},\underline{\theta}\right)$. Let the second-best quantities, $\overline{q}^{SB}$ and $\underline{q}^{SB}$, be the ones that solve the above problem.

(b) Show, by solving as much as you need of the problem, how $\overline{q}^{SB}$ relates to $\overline{q}^{FB}$. Moreover, show that $\underline{q}^{SB} \geq \underline{q}^{FB}$. You are allowed to assume that the second-order condition is satisfied (and you will not get any credit if you nevertheless investigate that).

(c) In the course we also showed that $\underline{q}^{SB} = \underline{q}^{FB}$. *Explain, in words, the idea behind the proof that we used*. You will not get any credit for actually doing the math, but you should explain verbally all the steps of the proof and why they can help us show the result $\underline{q}^{SB} = \underline{q}^{FB}$. Imagine that you are explaining to another person—who has successfully solved part (b) but does not know how to prove that $\underline{q}^{SB} = \underline{q}^{FB}$—how he/she can do it. After your explanation, this person should be able to follow your description, add the math calculations, and thereby show the result.

Question 2: Sharecropping with a continuum of effort and output levels

This is a model of sharecropping that lets both the farmer's effort choice and the output level be continuous. It builds on a similar model from the course, which assumed a binary effort choice and a binary output level.

A landlord (the principal of the model) owns a piece of land and wants to lease the land to a poor farmer (the agent). If entering such an agreement, the farmer will, when farming the land, choose what effort to make, $e \in [0, \infty)$. The associated effort cost (which enters additively in the payoff function) equals ce, where $c \in (0, 1)$. Depending on the farmer's effort and on the weather, an output level $q \in [0, 1]$ is realized. The cumulative distribution function that maps the effort e into an output level q is given by $F(q) = q^e$; the associated density is denoted by $f(q)$. The market price of the output equals unity. Hence, q is also the market value of the output.

The landlord (and the court) can observe which output that is realized but not what effort the farmer has chosen. Therefore, in principle, the contract between the landlord and the farmer could consist of any function that indicates how much the farmer should pay the landlord in each possible output state. However, the contract that is actually used is a so-called sharecropping contract, which is characterized by a single number, $\alpha \in [0, 1]$. The number α is the share of output that the farmer is allowed to *keep*, whereas the remaining share $1 - \alpha$ is paid to the landlord. The realized profit of the landlord, who is risk neutral, therefore equals $V = (1 - \alpha) q$, and the expected profit equals

$$EV = (1 - \alpha) \int_0^1 q f(q)\, dq = \frac{(1 - \alpha) e}{1 + e}.$$

The farmer is also risk neutral and her payoff is given by $U = \alpha q - ce$. In expected terms, this becomes

$$EU = \alpha \int_0^1 q f(q)\, dq - ce = \frac{\alpha e}{1 + e} - ce.$$

The farmer's outside option would yield the payoff zero. It is assumed that the landlord has all the bargaining power and makes a take-it-or-leave-it offer to the farmer.

(a) Characterize the second-best optimal value of α, using the first-order approach.[2] Assume that the second-order conditions are satisfied (you will not get any credit if you nevertheless investigate whether these conditions hold).

Now, instead of the model discussed above, consider the following moral hazard model with a risk neutral principal and a risk neutral agent who is protected by limited liability. There are two effort levels (0 and 1) and five output levels (y_1, y_2, y_3, y_4, and y_5). The probabilities with which the different output levels realize, given the two different effort levels, are indicated in the following table:

	Effort = 0	Effort = 1
y_1	$\pi_{10} = 0.2$	$\pi_{11} = 0.2 - 2x_B$
y_2	$\pi_{20} = 0.2$	$\pi_{21} = 0.2 - x_A$
y_3	$\pi_{30} = 0.2$	$\pi_{31} = 0.2 + x_A$
y_4	$\pi_{40} = 0.3$	$\pi_{41} = 0.3 + x_B,$
y_5	$\pi_{50} = 0.1$	$\pi_{51} = 0.1 + x_B,$

where $x_A \in [0, 0.1]$ and $x_B \in [0, 0.1]$.

(b) What is the condition that we need to impose on the model to ensure that the principal's optimal contract is such that the agent's payment is strictly increasing in the level of output that is realized? For what values of x_A and x_B is this condition satisfied? Explain the intuition for why the condition matters.

End of Exam

[2] It suffices to derive an equality that implicitly defines α, as long as the equality contains no other endogenous variable. That is, solving for a closed-form expression for the optimal α is not required.