Written Exam for the M.Sc. in Economics Winter 2014-15

# Contract Theory

Final Exam / Master's Course

January 10, 2015

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

**This exam paper consists of three pages in total, including this one**

# Regular Exam, Fall 2014
# Contract Theory, January 10, 2015

*Attempt both questions.*
*Explain all the steps of your analysis and define any new notation that you use.*

## Question 1: Adverse selection and optimal procurement

This adverse selection model with two types is identical to one that we studied in the course.

A firm (the agent, $A$) interacts with a government procurement agency (the principal, $P$). $A$ produces office material that $P$ wants to purchase. $A$'s cost of producing $q$ units of office material is given by the function $C(q, \theta)$, where $\theta$ is an efficiency parameter. This function satisfies

$$C(0, \theta) = 0, \qquad C_q > 0, \qquad C_{qq} \geq 0,$$
$$C_\theta > 0, \qquad C_{q\theta} > 0, \qquad C_{qq\theta} \geq 0.$$

The value for $P$ of receiving $q$ units of office material is given by the function $S(q)$, which satisfies

$$S'(q) > 0, \qquad S''(q) < 0, \qquad S(0) = 0.$$

The efficiency parameter $\theta$ can take two values: $\theta \in \{\underline{\theta}, \overline{\theta}\}$, with $0 < \underline{\theta} < \overline{\theta}$. $A$ knows the value of $\theta$ perfectly. However, $P$ only knows that

$$\Pr[\theta = \underline{\theta}] = \nu \qquad \text{and} \qquad \Pr[\theta = \overline{\theta}] = 1 - \nu,$$

with $0 < \nu < 1$. The procurement agency has all the bargaining power and makes a take-it-or-leave-it offer to the firm. A contract can specify the quantity $q$ that $A$ must produce and deliver and the payment $t$ that $A$ will receive. Suppose that $P$ wants to offer different contracts to the two types of firms. $P$ is risk neutral and its payoff, given a quantity $q$ and a payment $t$, equals

$$V = S(q) - t.$$

$A$ is also risk neutral and its payoff, given a quantity $q$ and a payment $t$, equals

$$U = t - C(q, \theta).$$

$A$'s outside option (the same for both types) would yield the payoff zero.

$P$ offers a menu of two distinct contracts to $A$. As in the course, the contract variables are indicated either with "upper-bars" or with "lower-bars", depending on which type the contract is aimed at. $P$'s problem is to choose $(\underline{t}, \underline{q}, \overline{t}, \overline{q})$ so as to maximize

$$\nu\left[S(\underline{q}) - \underline{t}\right] + (1 - \nu)\left[S(\overline{q}) - \overline{t}\right]$$

subject to the following four constraints:

$$\overline{t} - C(\overline{q}, \overline{\theta}) \geq 0, \tag{IR-bad}$$

$$\underline{t} - C(\underline{q}, \underline{\theta}) \geq 0, \tag{IR-good}$$

$$\overline{t} - C(\overline{q}, \overline{\theta}) \geq \underline{t} - C(\underline{q}, \overline{\theta}), \tag{IC-bad}$$

$$\underline{t} - C(\underline{q}, \underline{\theta}) \geq \overline{t} - C(\overline{q}, \underline{\theta}). \tag{IC-good}$$

**(a)** Explain in words what each one of the four constraints says and why it must be satisfied.

**(b)** Prove that incentive compatibility and Spence-Mirrlees ($C_{q\theta} > 0$) imply monotonicity; that is, show that if the inequalities defining incentive compatibility hold and if the Spence-Mirrlees condition is satisfied, then the quantity offered to the $\underline{\theta}$-type agent is at least as large as the one offered to the $\overline{\theta}$-type agent.

**(c)** The first best optimal quantities are defined by $S'(\underline{q}^{FB}) = C_q(\underline{q}^{FB}, \underline{\theta})$ and $S'(\overline{q}^{FB}) = C_q(\overline{q}^{FB}, \overline{\theta})$, respectively. Assume that the constraints (IR-good) and (IC-bad) are lax at the second-best optimum (so that they can be disregarded). Show that, at the second-best optimum, the good type's quantity is *not* distorted relative to the first best ($\underline{q}^{SB} = \underline{q}^{FB}$) and that the bad type's quantity is distorted downwards ($\overline{q}^{SB} < \overline{q}^{FB}$).

**(d)** Explain the intuition for the results you were asked to show under (c). Also explain the nature of the trade-off that the principal faces.

## Question 2: Moral hazard and insurance when there are three outcomes

The following is a model of moral hazard in an insurance market, with three possible outcomes. The principal ($P$) is a monopoly insurance company. The agent ($A$) is a home owner who may want to purchase a fire insurance. $A$ can choose whether to "be careful" ($e = 1$) and incur a cost $\psi > 0$ or "not be careful" ($e = 0$) and incur no cost. This choice cannot be observed by $P$.

Depending on whether $A$ is careful or not and on whether she is lucky or unlucky, there may be no fire (N), a small fire (S) or a big fire (B). Given $A$'s choice $e \in \{0, 1\}$, the outcome $i \in \{N, S, B\}$ is realized with probability

$$\Pr[i \mid e] = \pi_{ie} > 0,$$

with $\pi_{N0} + \pi_{S0} + \pi_{B0} = 1$ and $\pi_{N1} + \pi_{S1} + \pi_{B1} = 1$. The following conditions are assumed to hold:

$$\pi_{B1} \leq \pi_{B0} \text{ and } \pi_{B1} + \pi_{S1} \leq \pi_{B0} + \pi_{S0}, \quad \text{(FOSD)}$$

with at least one of the inequalities holding strictly.

$A$'s disutility of having a small (respectively, big) fire, measured in monetary terms as a deduction from her income, is denoted by $d_S$ (respectively, by $d_B$). It is assumed that $d_B > d_S > 0$, so both kinds of fire are costly but a big fire is most costly. One can show that, under the assumption (FOSD), the expected loss is smaller if $A$ makes an effort.[1] The insurance premium is denoted by $p$. The indemnity paid to $A$ in case of a small (respectively, big) fire is denoted by $a_S$ (respectively, by $a_B$). $A$'s monetary income is denoted by $w > 0$. All in all, this means that $A$'s utility if purchasing the insurance is as follows:

$$\begin{cases} u(w - p) \stackrel{\text{def}}{=} u_N & \text{if no fire} \\ u(w - p - d_S + a_S) \stackrel{\text{def}}{=} u_S & \text{if a small fire} \\ u(w - p - d_B + a_B) \stackrel{\text{def}}{=} u_B & \text{if a big fire,} \end{cases}$$

where $u' > 0$ and $u'' < 0$. The shorthand notation $u_N$, $u_S$ and $u_B$ defined above will be convenient when solving the model. $A$'s utility from her outside option if not buying an insurance is given by $u(\widehat{w})$, where $\widehat{w}$ is a certainty equivalent defined implicitly by the following equality:[2]

$$u(\widehat{w}) = \pi_{N1} u(w) + \pi_{S1} u(w - d_S) + \pi_{B1} u(w - d_B) - \psi.$$

Suppose $P$ wants to induce $A$ to make an effort. Also, let $h$ denote the inverse of $A$'s utility function $u$ (thus $h' > 0$ and $h'' > 0$). We can then write $P$'s expected profits as follows:

$$\pi = \widehat{w} - \pi_{N1} h(u_N) - \pi_{S1} h(u_S) - \pi_{B1} h(u_B),$$

where $\widehat{w} \stackrel{\text{def}}{=} w - \pi_{B1} d_B - \pi_{S1} d_S$. $P$ wants to maximize these profits w.r.t. $u_N$, $u_S$ and $u_B$, subject to the following two constraints:

$$\pi_{N1} u_N + \pi_{S1} u_S + \pi_{B1} u_B - \psi \geq u(\widehat{w}), \quad \text{(IR-H)}$$

$$(\pi_{N1} - \pi_{N0}) u_N + (\pi_{S1} - \pi_{S0}) u_S + (\pi_{B1} - \pi_{B0}) u_B \geq \psi. \quad \text{(IC)}$$

**(a)** Show that IR-H and IC bind at the optimum.

**(b)** Show that, at the optimum, the relationship $u_S \geq u_B$ holds if, and only if, the following condition is satisfied:

$$\frac{\pi_{B0}}{\pi_{B1}} \geq \frac{\pi_{S0}}{\pi_{S1}} \quad \text{(MLRP)}$$

**(c)** One can show that also if, as assumed in the model, the condition (FOSD) is satisfied, the condition (MLRP) may be violated. This means that there exist parameter values for which, at the optimal contract, we have $u_B > u_S$; that is, $A$ gets a *higher* utility after a big fire than after a small fire. Explain the intuition for why it can be optimal for $P$ to design a contract with this feature.

**(d)** Show, formally, that there cannot be full insurance at the optimum. Also, provide verbal arguments for why, or why not, the optimal contract involves full insurance if $P$ induces $e = 0$.

# End of Exam

[1] That is, $\pi_{B1} d_B + \pi_{S1} d_S < \pi_{B0} d_B + \pi_{S0} d_S$.

[2] We here assume that $A$, when being uninsured, prefers $e = 1$.