Written Exam for the M.Sc. in Economics Winter 2014-15

Contract Theory

Final (Resit) Exam / Master's Course

February 19, 2015

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

This exam paper consists of four pages in total, including this one

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Attempt both questions.
Explain all the steps of your analysis and define any new notation that you use.

Question 1: Moral hazard with mean-variance preferences

Consider the following moral hazard model with mean-variance preferences that we studied in the course. There is one (single) agent, A, and one principal, P. A chooses an effort level $e \in \Re_+$, thereby incurring the cost $c(e) = \frac{1}{2}e^2$. Given a choice of e, the output (i.e., A's performance) equals $q = e + z$, where z is an exogenous random term drawn from a normal distribution with mean zero and variance ν. It is assumed that P can observe q but not e. Moreover, neither P nor A can observe z. A's wage (i.e., the transfer from P to A) can only be contingent on the output q. It is restricted to be linear in q:

$$t = \alpha + \beta q = \alpha + \beta (e + z).$$

A is risk averse and has a CARA utility function: $U = -\exp[-r(t - c(e))]$, where r (> 0) is the coefficient of absolute risk aversion. Therefore A's expected utility is

$$EU = -\int_{-\infty}^{\infty} \exp[-r(t - c(e))] f(z)\, dz,$$

where $f(z)$ is the density of the normal distribution. P's objective function is

$$V = q - t = q - \alpha - \beta q = (1 - \beta)(e + z) - \alpha,$$

which in expected terms becomes $EV = (1 - \beta)e - \alpha$. It is also assumed that A's outside option utility is $\widehat{U} = -\exp\left[-r\widehat{t}\right]$, where $\widehat{t} > 0$. The timing of events is as follows.

1. P chooses the contract parameters, α and β.

2. A accepts or rejects the contract and, if accepting, chooses an effort level.

3. The noise term z is realized and A and P get their payoffs.

Answer the following questions:

(a) Solve for the β parameter in the second-best optimal contract, denoted by β^{SB} (you do not need to solve for α^{SB}, and you will not get any credit if you nevertheless do that). You should make use of the following (well-known) result:

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

[You are encouraged to attempt parts (b)–(d) also if you have not been able to answer part (a).]

(b) Does the agent get any rents at the second-best optimum? Do not only answer yes or no, but also explain how you can tell.

(c) The first-best values of the effort level and the β parameter equal $e^{FB} = 1$ and $\beta^{FB} = 0$, respectively. How do these values relate to the corresponding second-best values? In particular, is there under- or overprovision of effort at the second-best optimum?

(d) Consider the limit case where $r \to 0$. Explain what happens to the relationship between the second-best and the first-best effort levels. Also explain the intuition for this result.

Question 2: Consumer learning in an insurance market

The following is a model of an insurance market with adverse selection.[1] The principal (P) is a monopoly insurance company and the agent (A) is a car owner who may want to purchase a car insurance. Such an insurance compensates A for her financial loss in case the car is stolen. This loss is denoted by $d > 0$, and her income is denoted by $w > d$. Moreover, let p denote A's payment *to* P in case there is no theft; and let a denote the *net* compensation A *receives* from P in case the car is indeed stolen. A is risk averse and her utility function is denoted by u (where $u' > 0$ and $u'' < 0$). Thus, A's utility if purchasing the insurance is

$$\begin{cases} u(w-d+a) & \text{if car is stolen} \\ u(w-p) & \text{if car is not stolen.} \end{cases}$$

P is assumed to be a risk neutral profit maximizer.

The probability that a theft occurs, θ, can take two values: $\theta \in \{\underline{\theta}, \overline{\theta}\}$, with $0 < \underline{\theta} < \overline{\theta} < 1$. Initially, neither P nor A knows the value of θ: they both believe that $\Pr[\theta = \underline{\theta}] = \nu$ and $\Pr[\theta = \overline{\theta}] = 1 - \nu$, with $0 < \nu < 1$. However, A can, if incurring a cost $c > 0$, learn the value of θ. The full sequence of events is as follows.

(i) P commits to a menu of insurance policies, $\{(\underline{p}, \underline{a}), (\overline{p}, \overline{a})\}$, where the policy $(\underline{p}, \underline{a})$ is aimed at the $\underline{\theta}$ type and the policy $(\overline{p}, \overline{a})$ is aimed at the $\overline{\theta}$ type.

(ii) A observes the menu and then makes a choice whether or not to gather information, $x \in \{0, 1\}$. If $x = 1$, A must incur a cost $c > 0$ but receives a signal that perfectly reveals the true value of θ. If $x = 0$, A incurs no cost but does not obtain any new information about θ. The cost c enters A's payoff as an additive term. A's choice of x is not observed by P. Nor can P observe the signal that A receives if $x = 1$.

(iii) A decides whether to accept any insurance policy in the menu and, if so, which one.

Suppose P wants to induce A to gather information ($x = 1$). Also suppose that the parameters of the model are such that it is optimal to interact with both types and to offer them distinct contracts.

When solving P's problem it will be more convenient to think of P as choosing the utility levels directly, instead of the contract variables. Thus introduce the following notation:

$$\overline{u}_N \stackrel{\text{def}}{=} u(w - \overline{p}), \quad \overline{u}_A \stackrel{\text{def}}{=} u(w - d + \overline{a}),$$
$$\underline{u}_N \stackrel{\text{def}}{=} u(w - \underline{p}), \quad \underline{u}_A \stackrel{\text{def}}{=} u(w - d + \underline{a}).$$

Also let h be the inverse of u (hence $h' > 0$ and $h'' > 0$). We can now write P's ex ante expected profit as follows:

$$\pi = \widehat{w} - \upsilon\left[(1-\underline{\theta}) h(\underline{u}_N) + \underline{\theta} h(\underline{u}_A)\right] - (1-\upsilon)\left[(1-\overline{\theta}) h(\overline{u}_N) + \overline{\theta} h(\overline{u}_A)\right],$$

where $\widehat{w} \stackrel{\text{def}}{=} w - \left[\upsilon\underline{\theta} + (1-\upsilon)\overline{\theta}\right] d$ is A's wealth net of the ex ante expected monetary loss associated with an accident. P's problem is to maximize π w.r.t. $(\underline{u}_N, \underline{u}_A, \overline{u}_N, \overline{u}_A)$, subject to the following seven constraints:

$$(1-\overline{\theta})\overline{u}_N + \overline{\theta}\overline{u}_A \geq \overline{U}^*, \tag{IR-high}$$

$$(1-\underline{\theta})\underline{u}_N + \underline{\theta}\underline{u}_A \geq \underline{U}^*, \tag{IR-low}$$

$$(1-\overline{\theta})\overline{u}_N + \overline{\theta}\overline{u}_A \geq (1-\overline{\theta})\underline{u}_N + \overline{\theta}\underline{u}_A, \tag{IC-high}$$

$$(1-\underline{\theta})\underline{u}_N + \underline{\theta}\underline{u}_A \geq (1-\underline{\theta})\overline{u}_N + \underline{\theta}\overline{u}_A, \tag{IC-low}$$

$$\begin{aligned} EU_{x=1} \stackrel{\text{def}}{=} & \left[(1-\underline{\theta})\underline{u}_N + \underline{\theta}\underline{u}_A\right] \\ & + (1-\upsilon)\left[(1-\overline{\theta})\overline{u}_N + \overline{\theta}\overline{u}_A\right] - c \\ & \geq \upsilon\underline{U}^* + (1-\upsilon)\overline{U}^*, \end{aligned} \tag{IR-ante}$$

$$\begin{aligned} EU_{x=1} \geq & \ \upsilon\left[(1-\underline{\theta})\underline{u}_N + \underline{\theta}\underline{u}_A\right] \\ & + (1-\upsilon)\left[(1-\overline{\theta})\underline{u}_N + \overline{\theta}\underline{u}_A\right], \end{aligned} \tag{IG-low}$$

$$\begin{aligned} EU_{x=1} \geq & \ \upsilon\left[(1-\underline{\theta})\overline{u}_N + \underline{\theta}\overline{u}_A\right] \\ & + (1-\upsilon)\left[(1-\overline{\theta})\overline{u}_N + \overline{\theta}\overline{u}_A\right], \end{aligned} \tag{IG-high}$$

where

$$\overline{U}^* \stackrel{\text{def}}{=} (1-\overline{\theta}) u(w) + \overline{\theta} u(w-d),$$

$$\underline{U}^* \stackrel{\text{def}}{=} (1-\underline{\theta}) u(w) + \underline{\theta} u(w-d)$$

are the two types' outside options.

(a) Explain briefly in words what each one of the seven constraints says and why the constraints are required if P wants to induce information gathering, interact with both types and offer them distinct contracts.

[1] It builds on a model that we studied in the course, but here the information structure is endogenous.

One can show that the constraints IC-low, IC-high, IR-high and IR-ante are implied by the other constraints. The Lagrangian of P's profit maximization problem can thus be written as

$$\begin{aligned}
\mathcal{L} = \widehat{w} - \upsilon \left[(1-\underline{\theta})\, h\left(\underline{u}_N\right) + \underline{\theta} h\left(\underline{u}_A\right) \right] \\
- (1-\upsilon) \left[\left(1-\overline{\theta}\right) h\left(\overline{u}_N\right) + \overline{\theta} h\left(\overline{u}_A\right) \right] \\
+ \lambda \left[(1-\underline{\theta})\, \underline{u}_N + \underline{\theta} \underline{u}_A - \underline{U}^* \right] \\
- \overline{\mu} \left\{ \upsilon \left[(1-\underline{\theta}) \left(\overline{u}_N - \underline{u}_N\right) + \underline{\theta} \left(\overline{u}_A - \underline{u}_A\right) \right] + c \right\} \\
+ \underline{\mu} \left\{ (1-\upsilon) \left[\left(1-\overline{\theta}\right) \left(\overline{u}_N - \underline{u}_N\right) + \overline{\theta} \left(\overline{u}_A - \underline{u}_A\right) \right] - c \right\},
\end{aligned}$$

where $\lambda \geq 0$ is the shadow price associated with IR-low, $\overline{\mu} \geq 0$ is the shadow price associated with IG-high, and $\underline{\mu} \geq 0$ is the shadow price associated with IG-low.

(b) Show that IG-low and IR-low bind at the optimum.

(c) Show that the $\underline{\theta}$ type is underinsured ($\underline{u}_N > \underline{u}_A$) at the optimum.

One can further show that if it is optimal for P to induce information gathering, then, at the optimum, IG-high is lax (i.e., $\overline{\mu} = 0$) and the $\overline{\theta}$ type is fully insured (i.e., $\overline{u}_N = \overline{u}_A$).

(d) Suppose that it is indeed optimal for P to induce information gathering. Then what is the effect on P's profits, at the optimum, of an exogenous increase in the information gathering cost c? Will such an increase make P's (optimized) profits increase or decrease, or are the profits unaffected by a change in c? Do not show any calculations, but explain in words the reasoning behind your answer.

End of Exam