Written Exam at the Department of Economics

Summer 2018

Derivatives Pricing

Final Exam - Retake

August 23, 2018

3 hours, open book

<u>Answers in English only</u>

The exam consists of <u>5 pages</u> in total

If you fall ill during an examination at Peter Bangsvej, you must contact an invigilator in order to be registered as having fallen ill. In this connection, you must complete a form. Then you submit a blank exam paper and leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Be careful not to cheat at exams!

You cheat at an exam, if during the exam, you:

- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Guidelines:

- The exam is composed of 4 problems, each carrying an indicative weight.
- If you lack information to answer a question, please make the necessary assumptions.
- Please clearly state any assumptions you make.
- All answers must be justified.

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You have just been hired as a junior dealer on the options trading desk in a large Nordic investment bank. Your new managing director already has a number of problems he needs your help solving. Eager to show that you know your stuff (because you have followed the course in Derivatives Pricing), you immediately start working on the problems.

Problem 1 (25%)

One of the traders has just bought a 3-month at-the-money vanilla option straddle (i.e. a put option and a call option) on a non-dividend paying stock S. Assume the Black-Scholes assumptions are satisfied, interest rates are zero, and the spot price of the underlying stock is $S_0 = 100$.

The trader bought the straddle at an implied volatility of 25%. Assume the realized volatility turns out to be a constant 20% and the terminal stock price is $S_T = 120$.

a) What is the final PnL of the trade at expiration if the trader continuously delta-hedges it at a volatility of 20%?

b) What is the final PnL of the trade at expiration if the trader continuously delta-hedges it at a volatility of 25%?

c) What is the final PnL of the trade at expiration if the trader does not delta-hedge at all?

d) Discuss why a delta-hedged vanilla option straddle is not a clean bet on volatility.

In reality, market participants cannot trade at the mid-price of the stock; they have to buy at the offer price and sell at the bid price. For simplicity, let us assume the trader must pay a transaction cost equal to 0.5% of the value of the shares traded every time the trader delta-hedges, no matter if he buys or sells the stock.

e) What is the final PnL of the trade at expiration if the trader continuously delta-hedges it at its implied volatility of 25%?

f) Assuming she decides only to rebalance her delta hedge weekly, approximately at what price should the trader have bought the straddle to account for the transaction costs? (*Hint: you may use the PDE model of Hoggard, Whalley, and Willmott, p.125-129 in the course book.)*

Problem 2 (30%)

To price its equity options, the competitor bank uses a simple local volatility model. Assuming interest rates and dividends are zero, the risk-neutral process takes the form

$$\frac{dS_t}{S_t} = \sigma(S_t)dW_t$$

where the local volatility function is defined by

$$\sigma(S_t) = v{S_t}^{\beta-1}$$

with $v > 0$ and $0 \leq \beta \leq 1$ being constant parameters.

Your managing director fears that the rival bank has a competitive advantage, as the option-pricing model is more flexible than your current in-house model. Therefore, he asks you to analyze the model and evaluate its impact on implied volatility.

a) In general, what are the advantages of using a local volatility model for equity options over the classic BSM model?

b) Show that the BSM model is a special case of the model above.

One can derive the following short-term approximation of implied volatility in terms of the local volatilities between the spot price and strike

$$\frac{\ln\left(\frac{K}{S}\right)}{\Sigma} = \int_0^{\ln\left(\frac{K}{S}\right)} \frac{1}{\sigma(x)}dx$$

c) Using this short-term approximation, derive an explicit expression for the BSM implied volatility Σ in this model. (*Hint: the local volatility function can be reparametrized as* $\sigma(x) = vS^{\beta-1}e^{(\beta-1)x}$ *where* $x = \ln\left(\frac{K}{S}\right)$ *is the log-moneyness.*)

d) What is the implied volatility in the at-the-money limit $\frac{K}{S} \to 1$?

Taylor expanding the terms, one can show that the implied volatility approximation in Question **c)** can be simplified to

$$\Sigma \approx \frac{v}{S^{1-\beta}}\left[1 - \frac{1-\beta}{2}\ln\left(\frac{K}{S}\right)\right]$$

e) Based on this approximation, find the implied volatility skew $\frac{\partial\Sigma}{\partial K}$ and determine if the skew is flat, positive or negative for $\beta = 0$ and $\beta = 1$.

f) By replication, derive the price of a digital put option on S_t and explain how it depends on β.

g) Devise a trading strategy that profits if the implied volatility skew flattens but is otherwise protected against parallel level shifts in implied volatility.

Problem 3 (25%)

Your boss suggests extending the local volatility model you considered in the previous problem with stochastic volatility. Still assuming zero interest rates and dividends, you consider the model with risk-neutral dynamics:

$$\frac{dS_t}{S_t} = v_t {S_t}^{\beta-1} dW_t$$

$$dv_t = \epsilon v_t dZ_t$$

where $\epsilon > 0$ and $0 \leq \beta \leq 1$.

Moreover, it is assumed that $dW_t dZ_t = \rho dt = 0$, i.e., the Brownian motions driving the stock price and its instantaneous volatility are uncorrelated.

- **a)** Argue whether the model can capture the implied volatility skew observed in equity options markets when $\rho = 0$.
- **b)** Explain what is meant by the 'leverage effect' and determine if the model accommodates this effect.
- **c)** Find the distribution of the instantaneous variance v_t^2 and determine $E[v_t^2]$ and $var[v_t^2]$.
- **d)** Given the shape of the implied volatility surface for equity options, discuss if the distribution you found in Question **c)** is a realistic distribution of the stock's variance as time-to-expiry increases.

Suddenly, an American hedge fund calls you up and asks for a price of an exotic derivative contract with the following payoff at expiration

$$\max(e^{S_T} - e^K, 0)$$

for fixed strike K.

- **e)** Show how to statically replicate this exotic contract by a portfolio of vanilla call options on S_t.

Now, the spot price is $S_0 = 5$, assume $\beta = 0$, and let us substitute the v_t process with a simple two-state volatility process for which the average path volatility $\bar{v}$ takes only two values: 0.20 with 70% probability or 0.40 with 30% probability, independent of the stock price.

- **f)** Determine the price of the exotic contract with 1 year to expiration and strike $K = 5$.
  (*Hint: apply the 'mixing theorem', p.351-352 in the course book.)*

Problem 4 (20%)

In this last problem, you are asked to price a vanilla put option on the stock of a high-risk corporate. The stock price sometimes jumps and you therefore assume the following risk-neutral jump-diffusion dynamics of the log stock price

$$d \log S_t = \mu' dt + \sigma dW_t + J dq_t$$

where dq_t is a Poisson process and J is the jump size. A jump happens, on average, every 4 months, and when it does the log stock price decreases by 0.10. In the absence of jumps, the stock diffuses with a constant volatility of 15%. The current spot price is 100. Assume that the riskless rate and dividends are zero.

a) What is the probability of observing 3 jumps in a year?

b) Calculate the price of a 3-month put with strike 90, truncating the sum in the pricing formula to 3 jumps. (*Hint: the pricing formula for a call option can be found at p.404 in the course book, but remember here you are asked to price a put option)*

c) Argue whether this price is smaller, larger or the same in the BSM model with the same constant diffusion volatility of 15%, everything else equal.

d) Is the implied volatility skew positive or negative in this jump-diffusion model?

e) When time-to-expiry increases, why does the volatility smile eventually flatten in a jump-diffusion model?

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