Written Exam at the Department of Economics

Summer 2018

Derivatives Pricing

Final Exam

May 31, 2018

3 hours, open book exam

<u>Answers only in English.</u>

This exam question consists of 5 pages in total

NB: If you fall ill during an examination at Peter Bangsvej, you must contact an invigilator in order to be registered as having fallen ill. In this connection, you must complete a form. Then you submit a blank exam paper and leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Be careful not to cheat at exams!

You cheat at an exam, if during the exam, you:

- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Guidelines:

- The exam is composed of 4 problems, each carrying an indicative weight.
- If you lack information to answer a question, please make the necessary assumptions.
- Please clearly state any assumptions you make.
- All answers must be justified.

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You have just been hired as an analyst on the options trading desk in a large Nordic investment bank. Your new boss has a number of problems he needs your help with. Eager to show the senior traders that you know your stuff (as you have followed the course in Derivatives Pricing), you immediately start working on the problems.

Problem 1 (25%)

One of the traders has just sold a 1-year vanilla call option with strike 90. Assume the Black-Scholes assumptions are satisfied, the current stock price $S_0 = 90$, interest rates and dividends are zero, and the stock volatility $\sigma = 20\%$. The trader sold the option at its BSM price and continuously delta-hedges it using an implied volatility of 20%.

a) If the instantaneous realized volatility is a constant 20% and the terminal stock price is $S_T = 90$, what is the final PnL (Proft & Loss) of the delta-hedged call option in one year?

Now, let us see what happens to the final PnL when we start relaxing the BSM assumptions. We first investigate what happens if realized volatility is no longer equal to the volatility used to calculate the delta hedge:

b) If the instantaneous realized volatility is a constant 10% and the terminal stock price is $S_T = 90$, what can you say about the final PnL of the delta-hedged call in one year?

c) If the instantaneous realized volatility is initially low and then high with the final root-mean square volatility of 20% and the terminal stock price is $S_T = 90$, what can you say about the final PnL of the delta-hedged call in one year?

d) Would the final PnL in questions **a), b) and c)** be any different if the final stock price turns out to be $S_T = 120$?

e) If the client does not delta-hedge the call option, explain under what scenarios (i.e., combinations of final stock price and realized volatility) you would both end up making money on the trade.

In practice, continuous delta hedging is usually not feasible. Instead, traders hedge their positions at discrete time intervals.

f) Why is continuous delta hedging not feasible in practice?

g) How would discrete delta hedging affect your conclusion in question **a)**?

Problem 2 (25%)

Assume the price of a non-dividend paying tradable asset X_t follows the risk-neutral dynamics

$$dX_t = \tilde{\sigma} dW_t$$

where $\tilde{\sigma}$ is a positive constant.

a) Why is this not a reasonable model for the price of a stock?

b) How is the interpretation of the volatility $\tilde{\sigma}$ different in this model compared to the volatility in the BSM model?

c) Show that you can rewrite this model as a local volatility model with risk-neutral dynamics given by

$$\frac{dX_t}{X_t} = \sigma(X_t) dW_t$$

d) Find an approximation of implied volatility as a function of the strike by using that implied volatility is roughly the average of the local volatilities between the current spot price and the strike.

e) Show that the implied volatility skew is negative in this model.

f) Discuss why equity index options usually exhibit a negative implied volatility skew.

To use the model for option pricing, you need to calibrate it to market prices. Therefore, you have obtained the following mid prices of call options on the asset X_t from your broker:

Strike	Expiry (years)	Call price
95	1.00	10.727
100	1.00	7.979
105	1.00	5.727
100	1.05	8.176

The current spot price is $X_0 = 100$ and let the riskless interest rate be zero.

g) By valuing a calendar and butterfly spread, approximate the at-the-money local volatility in one year using Dupire's equation and estimate the volatility $\tilde{\sigma}$.

Your colleague has just sold a 1-year at-the-money call option on X_t at an implied volatility of 20%. The current spot price is 100 and $\tilde{\sigma} = 20$.

h) If she delta-hedges the option using the BSM model based on the implied volatility, has she then over- or under-hedged the call option relative to the delta in the local volatility model?

Problem 3 (20%)

Assuming zero interest rates and dividends, the index price S_t is governed by the Heston model with risk-neutral dynamics

$$\frac{dS_t}{S_t} = \sqrt{V_t}dW_t$$
$$dV_t = -\lambda(V_t - \bar{v})dt + \eta\sqrt{V_t}dZ_t$$
$$dW_t dZ_t = \rho dt$$

where $V_0, \lambda, \bar{v}, \eta$, ρ are parameters of the model.

a) What is the impact of stochastic volatility on the implied volatility surface?

b) Explain how the speed of mean-reversion in volatility λ and the correlation ρ impact the implied volatility surface.

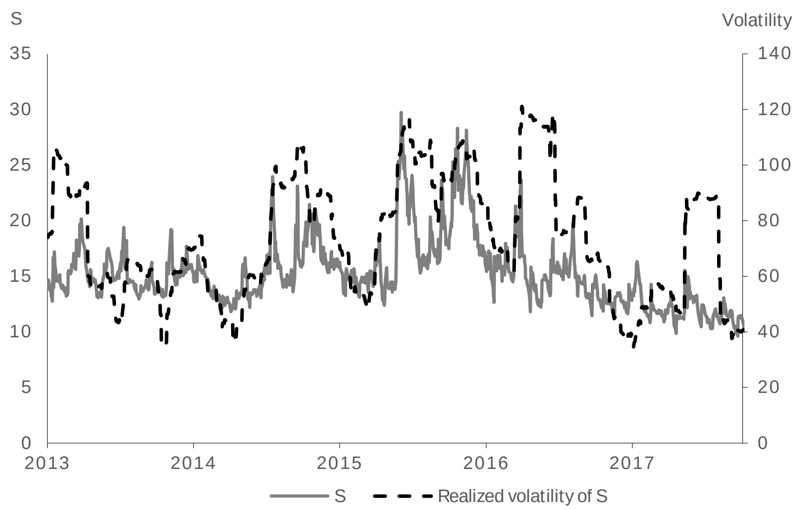

Figure 1: the historical prices of S_t and its 60-days' realized volatility

c) Based on the historical data on S_t in Figure 1, which sign would you expect ρ to have?

d) Why do stochastic volatility models have difficulties generating the steep short-term skew observed in equity options markets?

e) Devise a trading strategy in terms of vanilla options that will profit if volatility of volatility increases but is neutral to the overall level of volatility.

Problem 4 (30%)

Assume the Black-Scholes assumptions are satisfied. For simplicity, let the riskless interest rate be zero and assume the underlying stock pays zero dividends.

The BSM model assumes the risk-neutral dynamics of underlying asset follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \sigma dW_t$$

a) Derive the risk-neutral dynamics of $\log(S_t)$.

b) If the current stock price is 80 and the stock's volatility is 20% per year, what is the risk-neutral probability that it ends above 150 in 1 year?

One of the senior trader asks you to price a European-style exotic derivative contract to an important client. The payoff at expiration *T* of this contract is

$$\max(S_T^2 - K^2, 0)$$

where *K* is the strike and S_T is the terminal stock price

c) Derive an analytical formula for the contract's price in the BSM model.

d) If the current stock price is 80, *K*=100*,* the stock's volatility is 20% per year, and time to expiration is 1 year, what is the price of this contract in the BSM model?

Your senior colleague disagrees with your pricing of the exotic contract. She does not believe the BSM assumptions and argues that you need to use a more advanced model that fits the implied volatility smile.

e) Show that the exotic contract can be statically replicated with a portfolio of call options.

Assume you have calibrated two different stochastic volatility models; a model with mean-reversion in volatility and a model without mean-reversion in volatility. It turns out, however, that both models exactly fit the market prices of vanilla options with expiration at time *T* and therefore completely match the observed implied volatility smile.

f) Do the two models give the same or different prices of the exotic contract?

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