Financial Econometrics A | Final Exam | February 16th, 2018

Solution Key

Question A:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = \mu + \sqrt{\omega + \beta x_{t-1}^2} z_t \tag{A.1}$$

where the innovation z_t satisfies

$$z_t \sim i.i.d. N(0, 1). \tag{A.2}$$

The model parameters $\theta = (\mu, \beta, \omega)$ satisfy $\mu \in \mathbb{R}$, $\beta \geq 0$, and $\omega > 0$.

Question A.1: Provide conditions on $\theta = (\mu, \beta, \omega)$ such that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + x^2$.

Solution: Standard derivations from the lecture note yield that the drift criterion is satisfied if $\beta < 1$. Detailed arguments should be provided, including that x_t is a Markov chain with a positive and continuous transition density.

Question A.2: The log-likelihood contribution for the model is

$$l_t(\theta) = -\frac{1}{2}\left[\log(\omega + \beta x_{t-1}^2) + \frac{(x_t - \mu)^2}{\omega + \beta x_{t-1}^2}\right].$$

Suppose that the true values of β and ω are known such that $(\beta, \omega) = (\beta_0, \omega_0)$. This means that the only parameter to estimate is μ.
Based on a sample $(x_0, x_1, ..., x_T)$, show that the maximum likelihood estimator for μ is

$$\hat{\mu} = \frac{\sum_{t=1}^T x_t/(\omega_0 + \beta_0 x_{t-1}^2)}{\sum_{t=1}^T 1/(\omega_0 + \beta_0 x_{t-1}^2)}. \tag{A.3}$$

Solution: The result follows immediately by solving $\frac{\partial}{\partial \mu}\sum_{t=1}^T l_t(\theta) = 0$ for μ and setting $(\beta, \omega) = (\beta_0, \omega_0)$.

Question A.3: Assume that the true values $\omega_0 > 0$ and $\beta_0 > 0$ such that x_t is weakly mixing with $E[x_t^2] < \infty$.
Argue that for some constant $c > 0$,

$$E\left[\frac{z_t^2}{\omega_0 + \beta_0 x_{t-1}^2}\right] \leq c. \tag{A.4}$$

and that

$$\frac{1}{T}\sum_{t=1}^{T}\frac{z_t}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}} \xrightarrow{P} 0 \quad \text{as } T\to\infty.$$

Let μ_0 denote the true value of μ.
With $\hat{\mu}$ the maximum likelihood estimator for μ in (A.3), show that

$$\hat{\mu} \xrightarrow{P} \mu_0 \quad \text{as } T\to\infty.$$

Solution: Since $\left(\omega_0+\beta_0 x_{t-1}^2\right)^{-1} \leq \omega_0^{-1}$, it holds that $E\left[\frac{z_t^2}{\omega_0+\beta_0 x_{t-1}^2}\right] \leq E\left[\frac{z_t^2}{\omega_0}\right] = \omega_0^{-1}$. Next, observe that $\frac{z_t}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}} = \frac{(x_t-\mu_0)}{(\omega_0+\beta_0 x_{t-1}^2)} =: f(x_t, x_{t-1})$. Due to (A.4), $E[|f(x_t,x_{t-1})|]<\infty$. By the LLN for weakly mixing processes, $T^{-1}\sum_{t=1}^{T} f(x_t,x_{t-1}) \xrightarrow{p} E[f(x_t,x_{t-1})] = E\left[\frac{z_t}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}}\right]$. By the law of iterated expectations, $E\left[\frac{z_t}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}}\right] = E[z_t]E\left[\frac{1}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}}\right] = 0$. We conclude that $\frac{1}{T}\sum_{t=1}^{T}\frac{z_t}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}} \xrightarrow{P} 0$. Lastly, $\hat{\mu} = \frac{\sum_{t=1}^{T} x_t/(\omega_0+\beta_0 x_{t-1}^2)}{\sum_{t=1}^{T} 1/(\omega_0+\beta_0 x_{t-1}^2)} = \mu_0 + \frac{T^{-1}\sum_{t=1}^{T} z_t/(\omega_0+\beta_0 x_{t-1}^2)^{1/2}}{T^{-1}\sum_{t=1}^{T} 1/(\omega_0+\beta_0 x_{t-1}^2)}$. By the LLN for weakly mixing processes, $T^{-1}\sum_{t=1}^{T} 1/(\omega_0+\beta_0 x_{t-1}^2) \xrightarrow{P} E\left[1/(\omega_0+\beta_0 x_{t-1}^2)\right] < \infty$, and using that $\frac{1}{T}\sum_{t=1}^{T}\frac{z_t}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}} \xrightarrow{P} 0$, we conclude that $\hat{\mu} \xrightarrow{P} \mu_0$.

Question A.4: Maintaining the assumptions from Question A.3, with $\theta_0 = (\mu_0, \beta_0, \omega_0)$ the true value of θ, show that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t(\theta_0)}{\partial\mu} \xrightarrow{D} N(0,\Sigma) \quad \text{as } T\to\infty, \tag{A.5}$$

for some $\Sigma > 0$.
Explain briefly what the property in (A.5) can be used for.

Solution: We have that $\frac{\partial l_t(\theta_0)}{\partial\mu} = \frac{(x_t-\mu_0)}{\omega_0+\beta_0 x_{t-1}^2} =: f(x_t,x_{t-1})$. We show that (A.5) holds by applying the CLT for weakly mixing processes (that satisfy the drift criterion). From the previous question, $f(x_t,x_{t-1}) = \frac{z_t}{(\omega_0+\beta_0 x_{t-1}^2)^{1/2}}$, and we have that $E[f(x_t,x_{t-1})|x_{t-1}] = 0$ and $E[f^2(x_t,x_{t-1})]<\infty$, using the law of iterated expectations and (A.4). We conclude that (A.5) holds with $\Sigma = E[f^2(x_t,x_{t-1})] = E\left[\frac{z_t^2}{\omega_0+\beta_0 x_{t-1}^2}\right] = E[z_t^2]E\left[\frac{1}{\omega_0+\beta_0 x_{t-1}^2}\right] = E\left[\frac{1}{\omega_0+\beta_0 x_{t-1}^2}\right]$.

The property (A.5) is used for deriving the limiting distribution of the MLE. Ideally, a few comments about this is included. The distribution of the MLE is used when testing a hypothesis about the model parameters. One may also note that $\hat{\mu} - \mu_0 = \frac{T^{-1}\sum_{t=1}^{T} z_t/(\omega_0+\beta_0 x_{t-1}^2)^{1/2}}{T^{-1}\sum_{t=1}^{T} 1/(\omega_0+\beta_0 x_{t-1}^2)}$, so using $T^{-1}\sum_{t=1}^{T} 1/(\omega_0 + \beta_0 x_{t-1}^2) \xrightarrow{P} E\left[1/(\omega_0 + \beta_0 x_{t-1}^2)\right] = \Sigma$ and (A.5), it holds that $\sqrt{T}\,(\hat{\mu} - \mu_0) \xrightarrow{D} N(0, \Sigma^{-1})$.

Question A.5: For the model (A.1)-(A.2), the one-period Value-at-Risk (VaR) at risk level κ, $\text{VaR}_{T,1}^{\kappa}$, is

$$\text{VaR}_{T,1}^{\kappa} = -\mu - \sigma_{T+1}\Phi^{-1}(\kappa), \quad \kappa \in (0,1),$$

where $\sigma_{T+1}^2 = \omega + \beta x_T^2$ and where $\Phi^{-1}(\cdot)$ denotes the inverse CDF of the standard normal distribution.
With $(\omega, \beta) = (\omega_0, \beta_0)$ known, as in the previous questions, explain briefly how you would compute an estimate of $\text{VaR}_{T,1}^{\kappa}$, denoted $\widehat{\text{VaR}}_{T,1}^{\kappa}$.
Explain briefly how you would take into account the estimation uncertainty associated with $\widehat{\text{VaR}}_{T,1}^{\kappa}$.

Solution: Given an estimate of μ, $\hat{\mu}$, (see previous question) and known $(\omega, \beta) = (\omega_0, \beta_0)$ and $\Phi^{-1}(\kappa)$, one may estimate $\text{VaR}_{T,1}^{\kappa}$ as $\widehat{\text{VaR}}_{T,1}^{\kappa} = -\hat{\mu} - \sigma_{T+1}\Phi^{-1}(\kappa)$, where $\sigma_{T+1} = (\omega_0 + \beta_0 x_T^2)^{1/2}$. With μ_0 the true value of μ, the true VaR is $\text{VaR}_{T,1}^{\kappa} = -\mu_0 - \sigma_{T+1}\Phi^{-1}(\kappa)$. In order to assess the estimation uncertainty, we may note that $\widehat{\text{VaR}}_{T,1}^{\kappa} - \text{VaR}_{T,1}^{\kappa} = \mu_0 - \hat{\mu}$. Recall that under suitable conditions (discussed in the previous questions) $\sqrt{T}\,(\hat{\mu} - \mu_0) \xrightarrow{D} N(0, \Omega)$ for some Ω. In that case $\sqrt{T}(\widehat{\text{VaR}}_{T,1}^{\kappa} - \text{VaR}_{T,1}^{\kappa}) \xrightarrow{D} N(0, \Omega)$. This result can be used for obtaining a confidence band for the VaR estimate, as discussed in Assignment 2.

Question B:

Suppose that the logarithm of the price of a share of stock is given by

$$p(t) = \sigma W(t), \quad t \in [0, T], \tag{B.1}$$

where $\sigma > 0$ is constant and $W(t)$ is a Brownian motion.

Recall here that the Brownian motion $W(t)$ has the properties

1. $W(0) = 0$.
2. W has independent increments, i.e. if $0 \leq r < s \leq t < u$, then

   $$W(u) - W(t) \text{ and } W(s) - W(r)$$

   are independent.
3. The increments are normally distributed, i.e.

   $$W(t) - W(s) \sim N(0, t - s)$$

   for all $0 \leq s \leq t$.

Suppose that we have observed the price $p(t)$ at $n + 1$ equidistant points

$$0 = t_0 < t_1 < \ldots < t_n = T,$$

with

$$t_i = \frac{i}{n}T, \quad i = 0, ..., n.$$

Based on these points we obtain n log-returns given by

$$r(t_i) = p(t_i) - p(t_{i-1}), \quad i = 1, ..., n.$$

Question B.1: What is the distribution of $p(t)$?
Argue that $r(t_i)$ satisfies

$$r(t_i) \sim N\left(0, \sigma^2 \frac{T}{n}\right).$$

Show that

$$\text{cov}(r(t_i), r(t_{i-1})) = 0.$$

Solution: We have that $p(t) = \sigma[W(t) - W(0)] \sim N(0, \sigma^2 t)$. We have that $r(t_i) = \sigma[W(t_i) - W(t_{i-1})]$, such that $r(t_i) \sim N(0, \sigma^2(t_i - t_{i-1}))$ where $(t_i - t_{i-1}) = n/T$. Since the Brownian motion has independent increments, we have that $r(t_i) \sim I.I.D.N\left(0, \sigma^2 \frac{T}{n}\right)$, and in particular $\text{cov}(r(t_i), r(t_{i-1})) = 0$.

Question B.2: One way to measure the volatility of $p(T)$ would be to compute the realized volatility given by

$$RV(n) = \sum_{i=1}^{n} (r(t_i))^2 .$$

For a fixed $T > 0$, find the probability limit of $RV(n)$ as $n \to \infty$. Be precise about the arguments used for deriving the probability limit.
Give an interpretation of letting $n \to \infty$.

Solution: We may write $r(t_i) = \sigma\sqrt{T/n}\eta_i$, where $\eta_i \sim I.I.D.N(0,1)$. Hence by the LLN for independent processes, $RV(n) = \sigma^2 T n^{-1} \sum_{i=1}^{n} \eta_i^2 \xrightarrow{P} \sigma^2 T E[\eta_i^2] = \sigma^2 T$. Details should be provided.
The property $n \to \infty$ corresponds to increasing the sampling frequency over the fixed interval $[0, T]$ ("in-fill asymptotics").

Question B.3: Suppose that we do not observe the efficient log-price $p(t)$, but instead we observe $\tilde{p}(t)$ which is $p(t)$ contaminated by some noise $\tilde{\varepsilon}(t)$, that is

$$\tilde{p}(t) = p(t) + \tilde{\varepsilon}(t), \quad t \in [0, T],$$

with

$$\tilde{\varepsilon}(t) = \tilde{\sigma}\tilde{W}(t) + \mu t, \quad t \in [0, T],$$

where $\tilde{W}(t)$ is a Brownian motion and $\mu \in \mathbb{R}$ and $\tilde{\sigma} > 0$ are constants.
Now, the realized volatility measure $RV(n)$ from the previous question is infeasible due the fact that we do not observe $P(t)$. Instead we may compute

$$\widetilde{RV}(n) = \sum_{i=1}^{n} (\tilde{r}(t_i))^2 ,$$

where $\tilde{r}(t_i) = r(t_i) + \tilde{\varepsilon}(t_i) - \tilde{\varepsilon}(t_{i-1})$.
Assume that $W(t)$ and $\tilde{W}(t)$ are independent, that is $(W(t) : t \in [0, T])$ and $(\tilde{W}(t) : t \in [0, T])$ are independent. Similar to the previous question, for a fixed $T > 0$, derive the probability limit of $\widetilde{RV}(n)$ as $n \to \infty$. Compare with the probability limit of $RV(n)$.

Solution: Similar to the previous question, we may write $\tilde{\varepsilon}(t_i) - \tilde{\varepsilon}(t_{i-1}) = \mu T/n + \tilde{\sigma}\sqrt{T/n}\tilde{\eta}_i$ with $\tilde{\eta}_i \sim I.I.D.N(0,1)$. Hence, $\tilde{r}(t_i) = \sigma\sqrt{T/n}\eta_i + \mu T/n + \tilde{\sigma}\sqrt{T/n}\tilde{\eta}_i$, where the processes $(\eta_i : i = 1, ..., n)$ and $(\tilde{\eta}_i : i = 1, ..., n)$ are independent, as $(W(t) : t \in [0,T])$ and $(\tilde{W}(t) : t \in [0,T])$ are independent. Thus $\widetilde{RV}(n) = \sum_{i=1}^{n} \left[\sigma\sqrt{T/n}\eta_i + \mu T/n + \tilde{\sigma}\sqrt{T/n}\tilde{\eta}_i\right]^2$, and standard arguments yield that $\widetilde{RV}(n) \xrightarrow{P} T(\sigma^2 + \tilde{\sigma}^2)$. Details should be provided.

Question B.4: Figure 1 contains a plot of the realized volatility of the return of the Euro/Dollar exchange rate over 796 trading days. For each day, the realized volatility is based on $n = 47$ intra-daily return observations. Based on the figure and in light of your findings in the previous questions, do you think that the model $p(t) = \sigma W(t)$, from Question B.1 is suitable for the log-price of the exchange rate? Discuss briefly.

Solution: It is hard to say whether the model is suitable. Given that the model is correct, for n large, the realized volatility should be more or less constant over time, according to the previous questions. This does not seem to be the case based on the graph. Note, however, that no uncertainty about the estimates are reported. Also, from Question B.2 $RV(n) = \sigma^2 T n^{-1} \sum_{i=1}^{n} \eta_i^2 \sim T\sigma^2 n^{-1} \chi_n^2$ for any fixed n. Hence for any fixed n, the realized volatility should be an i.i.d. chi-squared-type process. This does not seem to be a good approximation of the observed RV which seems to exhibit some degree of persistence.

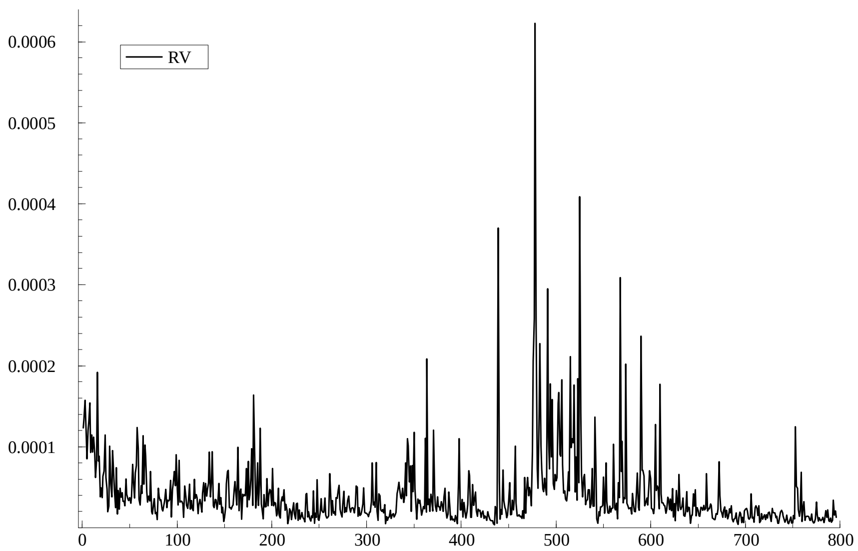

Figure 1: RV of Euro/Dollar returns