Written Exam at the Department of Economics winter 2017-18

Financial Econometrics A

Final Exam

Date: February 16th, 2018

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language for which you registered during exam registration.

This exam question consists of 6 pages in total

NB: If you fall ill during an examination at Peter Bangsvej, you must contact an invigilator in order to be registered as having fallen ill. In this connection, you must complete a form. Then you submit a blank exam paper and leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Please note there is a total of **9** questions that you should provide answers to. That is, **5** questions under *Question A*, and **4** under *Question B*.

Question A:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = \mu + \sqrt{\omega + \beta x_{t-1}^2} z_t \tag{A.1}$$

where the innovation z_t satisfies

$$z_t \sim i.i.d.N(0, 1). \tag{A.2}$$

The model parameters $\theta = (\mu, \beta, \omega)$ satisfy $\mu \in \mathbb{R}$, $\beta \geq 0$, and $\omega > 0$.

Question A.1: Provide conditions on $\theta = (\mu, \beta, \omega)$ such that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + x^2$.

Question A.2: The log-likelihood contribution for the model is

$$l_t(\theta) = -\frac{1}{2} \left[\log(\omega + \beta x_{t-1}^2) + \frac{(x_t - \mu)^2}{\omega + \beta x_{t-1}^2} \right].$$

Suppose that the true values of β and ω are known such that $(\beta, \omega) = (\beta_0, \omega_0)$. This means that the only parameter to estimate is μ.
Based on a sample $(x_0, x_1, ..., x_T)$, show that the maximum likelihood estimator for μ is

$$\hat{\mu} = \frac{\sum_{t=1}^{T} x_t / (\omega_0 + \beta_0 x_{t-1}^2)}{\sum_{t=1}^{T} 1/(\omega_0 + \beta_0 x_{t-1}^2)}. \tag{A.3}$$

Question A.3: Assume that the true values $\omega_0 > 0$ and $\beta_0 > 0$ such that x_t is weakly mixing with $E[x_t^2] < \infty$.
Argue that for some constant $c > 0$,

$$E\left[\frac{z_t^2}{\omega_0 + \beta_0 x_{t-1}^2}\right] \leq c. \tag{A.4}$$

and that

$$\frac{1}{T}\sum_{t=1}^{T} \frac{z_t}{(\omega_0 + \beta_0 x_{t-1}^2)^{1/2}} \xrightarrow{P} 0 \quad \text{as } T \to \infty.$$

Let μ_0 denote the true value of μ.
With $\hat{\mu}$ the maximum likelihood estimator for μ in (A.3), show that

$$\hat{\mu} \xrightarrow{P} \mu_0 \quad \text{as } T \to \infty.$$

Question A.4: Maintaining the assumptions from Question A.3, with $\theta_0 = (\mu_0, \beta_0, \omega_0)$ the true value of θ, show that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \frac{\partial l_t(\theta_0)}{\partial \mu} \xrightarrow{D} N(0, \Sigma) \quad \text{as } T \to \infty, \tag{A.5}$$

for some $\Sigma > 0$.
Explain briefly what the property in (A.5) can be used for.

Question A.5: For the model (A.1)-(A.2), the one-period Value-at-Risk (VaR) at risk level κ, $\text{VaR}_{T,1}^{\kappa}$, is

$$\text{VaR}_{T,1}^{\kappa} = -\mu - \sigma_{T+1}\Phi^{-1}(\kappa), \quad \kappa \in (0, 1),$$

where $\sigma_{T+1}^2 = \omega + \beta x_T^2$ and where $\Phi^{-1}(\cdot)$ denotes the inverse CDF of the standard normal distribution.
With $(\omega, \beta) = (\omega_0, \beta_0)$ known, as in the previous questions, explain briefly how you would compute an estimate of $\text{VaR}_{T,1}^{\kappa}$, denoted $\widehat{\text{VaR}}_{T,1}^{\kappa}$.
Explain briefly how you would take into account the estimation uncertainty associated with $\widehat{\text{VaR}}_{T,1}^{\kappa}$.

Question B:

Suppose that the logarithm of the price of a share of stock is given by

$$p(t) = \sigma W(t), \quad t \in [0, T], \tag{B.1}$$

where $\sigma > 0$ is constant and $W(t)$ is a Brownian motion.

Recall here that the Brownian motion $W(t)$ has the properties

1. $W(0) = 0$.
2. W has independent increments, i.e. if $0 \leq r < s \leq t < u$, then

   $$W(u) - W(t) \text{ and } W(s) - W(r)$$

   are independent.
3. The increments are normally distributed, i.e.

   $$W(t) - W(s) \sim N(0, t - s)$$

   for all $0 \leq s \leq t$.

Suppose that we have observed the price $p(t)$ at $n + 1$ equidistant points

$$0 = t_0 < t_1 < \ldots < t_n = T,$$

with

$$t_i = \frac{i}{n}T, \quad i = 0, ..., n.$$

Based on these points we obtain n log-returns given by

$$r(t_i) = p(t_i) - p(t_{i-1}), \quad i = 1, ..., n.$$

Question B.1: What is the distribution of $p(t)$?
Argue that $r(t_i)$ satisfies

$$r(t_i) \sim N\left(0, \sigma^2 \frac{T}{n}\right).$$

Show that

$$\text{cov}(r(t_i), r(t_{i-1})) = 0.$$

Question B.2: One way to measure the volatility of $p(T)$ would be to compute the realized volatility given by

$$RV(n) = \sum_{i=1}^{n} (r(t_i))^2 .$$

For a fixed $T > 0$, find the probability limit of $RV(n)$ as $n \to \infty$. Be precise about the arguments used for deriving the probability limit.
Give an interpretation of letting $n \to \infty$.

Question B.3: Suppose that we do not observe the efficient log-price $p(t)$, but instead we observe $\tilde{p}(t)$ which is $p(t)$ contaminated by some noise $\tilde{\varepsilon}(t)$, that is

$$\tilde{p}(t) = p(t) + \tilde{\varepsilon}(t), \quad t \in [0, T],$$

with

$$\tilde{\varepsilon}(t) = \tilde{\sigma}\tilde{W}(t) + \mu t, \quad t \in [0, T],$$

where $\tilde{W}(t)$ is a Brownian motion and $\mu \in \mathbb{R}$ and $\tilde{\sigma} > 0$ are constants.
Now, the realized volatility measure $RV(n)$ from the previous question is infeasible due the fact that we do not observe $P(t)$. Instead we may compute

$$\widetilde{RV}(n) = \sum_{i=1}^{n} (\tilde{r}(t_i))^2 ,$$

where $\tilde{r}(t_i) = r(t_i) + \tilde{\varepsilon}(t_i) - \tilde{\varepsilon}(t_{i-1})$.
Assume that $W(t)$ and $\tilde{W}(t)$ are independent, that is $(W(t) : t \in [0, T])$ and $(\tilde{W}(t) : t \in [0, T])$ are independent. Similar to the previous question, for a fixed $T > 0$, derive the probability limit of $\widetilde{RV}(n)$ as $n \to \infty$. Compare with the probability limit of $RV(n)$.

Question B.4: Figure 1 contains a plot of the realized volatility of the return of the Euro/Dollar exchange rate over 796 trading days. For each day, the realized volatility is based on $n = 47$ intra-daily return observations. Based on the figure and in light of your findings in the previous questions, do you think that the model $p(t) = \sigma W(t)$, from Question B.1 is suitable for the log-price of the exchange rate? Discuss briefly.

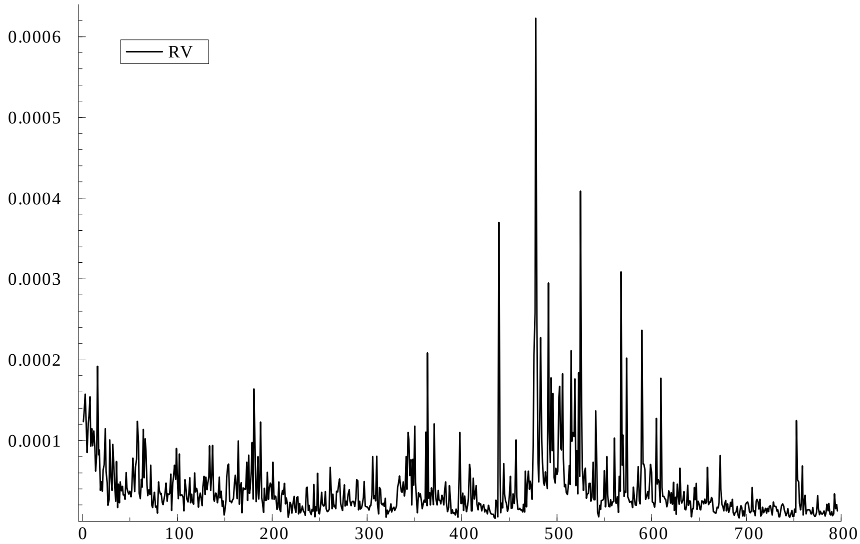

Figure 1: RV of Euro/Dollar returns