Written Exam at the Department of Economics winter 2017-18

Financial Econometrics A

Final Exam

Date: January 2nd, 2018

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language for which you registered during exam registration.

This exam question consists of 5 pages in total

NB: If you fall ill during an examination at Peter Bangsvej, you must contact an invigilator in order to be registered as having fallen ill. In this connection, you must complete a form. Then you submit a blank exam paper and leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Please note there is a total of **9** questions that you should provide answers to. That is, **4** questions under *Question A*, and **5** under *Question B*.

Question A:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = \sqrt{1 + \beta x_{t-1}^2} z_t \tag{A.1}$$

where the innovation z_t satisfies

$$z_t \sim i.i.d. N(0, \omega). \tag{A.2}$$

The model parameters $\theta = (\beta, \omega)$ satisfy $\beta \geq 0$ and $\omega > 0$.

Question A.1: Provide conditions on $\theta = (\beta, \omega)$ such that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + x^2$.

Question A.2: The log-likelihood contribution for the model is

$$l_t(\theta) = -\frac{1}{2}\left[\log(\omega + \omega\beta x_{t-1}^2) + \frac{x_t^2}{\omega + \omega\beta x_{t-1}^2}\right]. \tag{A.3}$$

With $\theta_0 = (\beta_0, \omega_0)$ the true value of θ, suppose that x_t is weakly mixing such that $E[x_t^2] < \infty$. Argue that for some constant $c > 0$,

$$E\left[\frac{x_t^2}{\omega_0 + \omega_0\beta_0 x_{t-1}^2}\right] \leq c.$$

Question A.3: Let $s_t(\theta)$ denote the first derivative in the direction ω of the log-likelihood contribution in (A.3), i.e.

$$s_t(\theta) = \frac{\partial l_t(\theta)}{\partial \omega}.$$

Let $\theta_0 = (\beta_0, \omega_0)$ be the vector of true parameter values. Note that $E[z_t^4] = 3\omega_0^2$.
With T the sample size, provide conditions such that as $T \to \infty$,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} s_t(\theta_0) \xrightarrow{D} N(0, \Sigma), \quad \Sigma = \frac{1}{2\omega_0^2} > 0. \tag{A.4}$$

Explain what (A.4) can be used for.

Question A.4: For the model (A.1)-(A.2), the one-period Value-at-Risk (VaR) at risk level κ, $\text{VaR}_{T,1}^{\kappa}$, is

$$\text{VaR}_{T,1}^{\kappa} = -\omega^{1/2}\sigma_{T+1}\Phi^{-1}(\kappa), \quad \kappa \in (0,1),$$

where $\sigma_{T+1}^2 = 1 + \beta x_T^2$ and where $\Phi^{-1}(\cdot)$ denotes the inverse CDF of the standard normal distribution.
Explain briefly how you would compute an estimate of $\text{VaR}_{T,1}^{\kappa}$.
Explain briefly how you would compute an estimate of the two-period VaR at risk level κ.

Question B:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = a_{s_t} x_{t-1} + \varepsilon_t, \tag{B.1}$$

where the error term ε_t satisfies

$$\varepsilon_t \sim i.i.d.N(0,1). \tag{B.2}$$

Moreover,

$$a_{s_t} = 1(s_t = 1)a_1 + 1(s_t = 2)a_2, \tag{B.3}$$

where s_t is a state variable that takes values in $\{1, 2\}$ according to the transition probabilities

$$P(s_t = j | s_{t-1} = i) = p_{ij}, \tag{B.4}$$

and $1(s_t = i) = 1$ if $s_t = i$ and $1(s_t = i) = 0$ if $s_t \neq i$ for $i = 1, 2$. We assume throughout that the processes (ε_t) and (s_t) are independent. The model parameters $\theta = (a_1, a_2)$ satisfy $a_1, a_2 \in \mathbb{R}$.

Question B.1: When is the process (s_t) weakly mixing?

Question B.2: Let $f(x_t | x_{t-1}, s_t)$ denote the conditional density of x_t given (x_{t-1}, s_t). Provide an expression for $f(x_t | x_{t-1}, s_t)$.

Question B.3: In the following we assume that $p_{11} = 1 - p_{22} =: p \in (0, 1)$ such that (s_t) is an i.i.d. process with $P(s_t = 1) = p$.
Show that

$$f(x_t | x_{t-1}) = f(x_t | x_{t-1}, s_t = 1)P(s_t = 1) + f(x_t | x_{t-1}, s_t = 2)P(s_t = 2) > 0.$$

Show that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + x^2$ if

$$a_1^2 p + a_2^2 (1 - p) < 1.$$

Question B.4: Maintaining the assumptions from Question B.3, we consider the log-likelihood function (up to a constant)

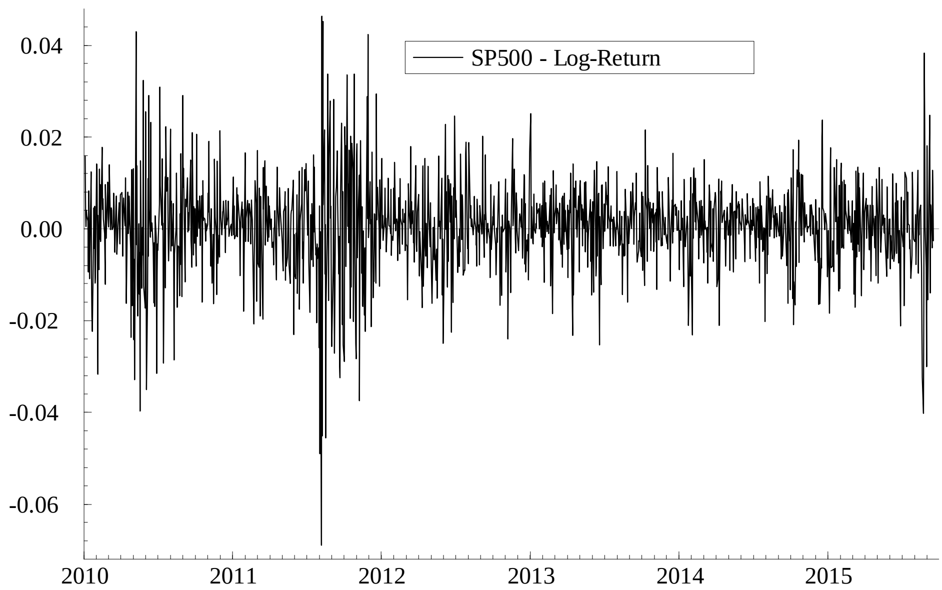
$$L_T(\theta) = \sum_{t=1}^{T} \Bigg[1(s_t = 1) \left\{ -\frac{(x_t - a_1 x_{t-1})^2}{2} + \log(p) \right\} + 1(s_t = 2) \left\{ -\frac{(x_t - a_2 x_{t-1})^2}{2} + \log(1 - p) \right\} \Bigg].$$

Show that the Maximum Likelihood Estimator for a_1 is

$$\hat{a}_1 = \frac{\sum_{t=1}^{T} 1(s_t = 1) x_t x_{t-1}}{\sum_{t=1}^{T} 1(s_t = 1) x_{t-1}^2}.$$

Assume that the joint process (s_t, x_{t-1}) is weakly mixing such that $E[x_{t-1}^2] < \infty$. Argue that $\hat{a}_1 \xrightarrow{P} a_1$ as $T \to \infty$.

Question B.5: The following figure shows the daily log-returns of the S&P 500 index for the period January 4, 2010 to September 17, 2015.



Discuss briefly whether the model in (B.1)-(B.4) is a reasonable model for the daily log returns of the S&P 500 index.