Written Exam for the M.Sc. in Economics 2014 (Fall Term)

Financial Econometrics A: Volatility Modelling

Final Exam: Masters course

January 19 2015 | 3-hour closed book exam.

Notes on Exam: Please note that there are a total of 9 questions which should all be answered.

Please note that the language used in your exam paper must correspond to the language of the title for which you registered during exam registration. I.e. if you registered for the English title of the course, you must write your exam paper in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title which was followed by "eksamen på dansk" in brackets, you must write your exam paper in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

Question 1:

Consider the ARCH_{abs} model for log-returns r_t as given by,

$$r_t = \sigma_t z_t \qquad \sigma_t^2 = a + b|r_{t-1}|,$$

with z_t iidN$(0,1)$ for $t = 1, ..., T$ and with r_0 fixed. Moreover, the parameters a and b are strictly positive, $a > 0$ and $b > 0$.

Q1.1:

Compare the model with the classic linear ARCH(1) model and discuss why it may be of interest in practice (if at all).

Q1.2:

Use the drift function $\delta(r) = 1 + r^2$, and show that for r^2 large,

$$E(\delta(r_t)|r_{t-1} = r) \leq \beta\delta(r),$$

for some $\beta < 1$ for any (positive) values of a and b, including the particular values $(a_0, b_0) = (1, 10)$. Argue that this implies that r_t is weakly mixing, and stationary with $Er_t^2 < \infty$ for any positive values of a and b.

Q1.3:

Next, turn to estimation of the parameter a (leaving b as fixed or known for simplicity). The Gaussian log-likelihood function $\ell_T(a)$ is given by,

$$\ell_T(a) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log\sigma_t^2 + \frac{r_t^2}{\sigma_t^2}\right).$$

Describe briefly how to obtain the MLE of a in a programming language such as "ox".

If we assumed z_t to be standardized $t_v(0,1)$ distributed instead, would optimization of $\ell_T(a)$ lead to the MLE? Explain why (or why not), and what one would obtain if it is not the MLE.

Q1.4:

A key condition for asympotic normality of the MLE $\hat{a}$ of a is that

$$\frac{1}{\sqrt{T}} \partial \ell_T(a) / \partial a|_{a=a_0, b=b_0} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left((r_t/\sigma_t)^2 - 1 \right) \frac{1}{\sigma_t^2} \xrightarrow{D} N(0, \phi),$$
$$\text{with } \phi/2 = E\left(\sigma_t^{-4}\right).$$

Show that this holds.

Show that $\phi < \infty$.

Please be specific about use of arguments and results you use and verify that they hold.

Q1.5:

We may conclude (do not show this) that

$$\sqrt{T}(\hat{a} - a_0) \xrightarrow{D} N\left(0, \phi^{-1}\right).$$

Explain how you would use this result in empirical work.

Sometimes the asymptotic result is reported on the form:

$$\sqrt{T}(\hat{\alpha} - a_0) \xrightarrow{D} N\left(0, \delta^{-1}\phi\delta^{-1}\right),$$

where $\delta \neq \phi$. Explain why this form is often used.

Question 2:

Q2.1:

Figure 2.1 shows a log-returns series x_t with $T = 1200$ observations together with the ACF for $|x_t|$. Furthermore, Table 2.1 shows some output from estimation of a GARCH(1,1) model with these data.

Based on Figure 2.1 and Table 2.1 explain if would you expect a GARCH(1,1) fit the data. Are there any ARCH effects in the residuals and is that a normal finding?

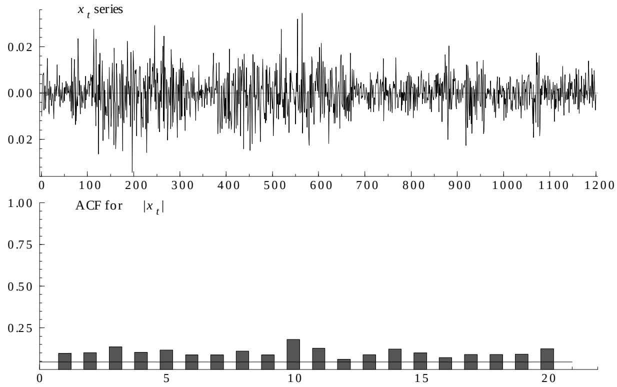

Figure 2.1

Table 2.1	
Parameter estimates in GARCH(1,1):	$\hat{\alpha}+\hat{\beta}=0.998$
Standarized residuals: $\hat{z}_t = x_t/\hat{\sigma}_t$	
Normality Test for $\hat{z}_t$:	p-value: 0.00
LM test for ARCH in $\hat{z}_t$:	p-value: 0.25

Q2.2:

Instead of GARCH(1,1) we apply a 2-state volatility model for x_t where:

$$x_t = \sigma_t z_t \text{ with}$$
$$\sigma_t^2 = \begin{cases} \sigma_2^2 & \text{if } s_t = 2 \\ \sigma_1^2 & \text{if } s_t = 1 \end{cases}$$

and z_t iidN$(0,1)$ distributed and $t = 1, ..., T$.

Assume initially that the variable s_t is **observed** with values correspond to high and low volatility regimes. For example, if $\sigma_1^2 < \sigma_2^2$, then $s_t = 1$ is the low, and $s_t = 2$ the high volatility regime.

Show that the MLE of $\hat{\sigma}_2^2$, is given by

$$\hat{\sigma}_2^2 = \frac{\sum_{t=1}^{T} 1\,(s_t = 2)\, x_t^2}{\sum_{t=1}^{T} 1\,(s_t = 2)}.$$

Q2.3:

Next, we assume that s_t is an **unobserved** two-state Markov-chain as given by the transition matrix,

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix},$$

with

$$p_{ij} = P(s_t = j | s_{t-1} = i)$$

such that $p_{11} = 1 - p_{12}$ and $p_{22} = 1 - p_{21}$.

Hence we wish to estimate $\theta = (\sigma_1^2, \sigma_2^2, p_{11}, p_{22})$. For this purpose let $f_t(i) = \log \sigma_i^2 + \frac{x_t{}^2}{\sigma_i^2}$ for $i = 1, 2$ and define the function $Q(\theta)$ by,

$$Q(\theta) = -\frac{1}{2} \sum_{t=1}^{T} \sum_{i=1,2} \left(p_t^*(i) f_t(i) - p_t^*(i,j) \log p_{ij} \right).$$

Here $p_t^*(j)$ are denoted *smoothed* probabilities and $p_t^*(i,j)$ *smoothed* transitition probabilities.

Explain what $f_t(1)$ is.

Explain what $p_t^*(1)$ can be used for.

Explain what $Q(\theta)$ is and how it may be used to find the MLE of θ.

Q2.4:

Discuss under which restrictions on the parameters in the transition matrix $\mathbf{P}$ you would expect the MLE of θ to be consistent and asymptotically Gaussian distributed.

From estimation one finds the MLE of p_{11} and p_{22} to be $\hat{p}_{11} = 0.98$ and $\hat{p}_{22} = 0.23$ respectively. Comment on these (make the necessary reservations about missing additional out-put).