Written Exam at the Department of Economics winter 2020-21

Financial Econometrics A

Final Exam

16 February, 2021

(3-hour open book exam)

Answers only in English.

The paper must be uploaded as <u>one PDF document</u>. The PDF document must be named with exam number only (e.g. '127.pdf') and uploaded to Digital Exam.

This exam question consists of 5 pages in total (including this front page).

This exam has been changed from a written Peter Bangsvej exam to a take-home exam with helping aids. Please read the following text carefully in order to avoid exam cheating.

Be careful not to cheat at exams!
You cheat at an exam, if you during the exam:

- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text. This also applies to text from old grading instructions.
- Make your exam answers available for other students to use during the exam
- Communicate with or otherwise receive help from other people
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Use parts of a paper/exam answer that you have submitted before and received a passed grade for without making use of source referencing (self plagiarism)

You can read more about the rules on exam cheating on the study information pages in KUnet and in the common part of the curriculum section 4.12.

Exam cheating is always sanctioned with a warning and dispelling from the exam. In most cases, the student is also expelled from the university for one semester.

Please note there is a total of **8** questions that you should provide answers to. That is, **4** questions under *Question A*, and **4** under *Question B*.

Question A:

Consider the ARCH model given by

$$y_t = \delta_t(\beta,\gamma)^{1/2} x_t, \; x_t = \sigma_t(\alpha) z_t,$$
$$\sigma_t^2(\alpha) = 1 - \alpha + \alpha x_{t-1}^2,$$
$$\delta_t(\beta,\gamma) = \beta + \gamma d(t), \; d(t) = \frac{t^2}{1+t^2},$$

with z_t iid $N(0,1)$ distributed, x_0 fixed and $t = 1, 2, ..., T$. Also $0 \leq \alpha < 1$, $\beta > 0$ and $\gamma \geq 0$.

As to the role of $\delta_t(\beta,\gamma)$, observe that $d(t) \in (0,1)$ and hence $\delta_t(\beta,\gamma) \in (\beta, \beta+\gamma)$.

Question A.1: Show that (using as usual the notation that δ_t and σ_t^2 denote $\delta_t(\beta,\gamma)$ and $\sigma_t^2(\alpha)$ respectively evaluated at the true values α_0, β_0 and γ_0),

$$E(y_t | x_{t-1}) = 0 \text{ and } V(y_t | x_{t-1}) = \delta_t \sigma_t^2$$

Show furthermore that

$$V(y_t) = \beta_0 + \gamma_0 d(t),$$

and state a sufficient condition on α_0 for this to hold.

Discuss the role of $\delta_t(\beta,\gamma)$. In particular discuss what happens if $\gamma_0 = 0$ and $\gamma_0 > 0$ respectively.

Question A.2: Fix all parameters at their true values, except γ and α. Show that the first order derivatives of the log-likelihood function is given by,

$$S_T^\gamma = \partial L(\alpha,\gamma)/\partial\gamma|_{\alpha=\alpha_0,\gamma=\gamma_0} = -\frac{1}{2}\sum_{t=1}^{T} s_t^\gamma, \quad s_t^\gamma = \frac{d(t)}{\delta_t}\left(1-z_t^2\right).$$

$$S_T^\alpha = \partial L(\alpha,\gamma)/\partial\alpha|_{\alpha=\alpha_0,\gamma=\gamma_0} = -\frac{1}{2}\sum_{t=1}^{T} s_t^\alpha, \quad s_t^\alpha = \frac{y_{t-1}^2/\delta_t}{1-\alpha_0+\alpha_0 y_{t-1}^2/\delta_t}\left(1-z_t^2\right).$$

Argue that $E(s_t^\gamma) = E(s_t^\alpha) = 0$.

Question A.3: Show that for $\alpha_0 \in (0,1)$,

$$T^{-1/2}\sum_{t=1}^{T} s_t^\alpha \xrightarrow{D} N(0,\xi\kappa) \quad \kappa = E\left(1-z_t^2\right)^2 = 3 \text{ and } \xi = E\left(\frac{y_{t-1}^2/\delta_t}{1-\alpha_0+\alpha_0 y_{t-1}^2/\delta_t}\right)^2.$$

Explain why $\xi < \infty$.

Question A.4: With $(\hat{\alpha},\hat{\gamma})$ the maximizer of the log-likelihood function, it can be shown that for $0 < \alpha_0 < 1$ and $\gamma_0 > 0$,

$$T^{1/2}\sum_{t=1}^{T}(\hat{\alpha}-\alpha_0, \hat{\gamma}-\gamma_0)$$

is asymptotically Gaussian. What would you expect the limiting distribution of the likelihood ratio statistic for $H : (\gamma = \gamma_0, \ \alpha = \alpha_0)$ is?

Question B:

Consider the model for $y_t \in \mathbb{R}$ given by

$$y_t = 1_{(s_t=1)} z_{t,1} + 1_{(s_t=2)} z_{t,2},$$

where

$$1_{(s_t=i)} = \begin{cases} 1 & \text{if } s_t = i \\ 0 & \text{if } s_t \neq i \end{cases}, \quad i = 1, 2,$$

and, for $i = 1, 2$, $(z_{t,i})$ is an i.i.d. process with

$$z_{t,i} \overset{d}{=} t_{v_i}, \quad i = 1, 2,$$

i.e. $z_{t,i}$ is Student's t-distributed with $v_i > 2$ degrees of freedom. The processes $(z_{t,1})$ and $(z_{t,2})$ are independent. Moreover, (s_t) is a two-state Markov chain with transition probabilities

$$P(s_t = j | s_{t-1} = i) = p_{ij} \in [0, 1], \quad i, j = 1, 2.$$

Assume throughout that (s_t) is independent of the processes $(z_{t,1})$ and $(z_{t,2})$.

Lastly, recall that if X is Student's t-distributed with $v > 0$ degrees of freedom, then the density of X is

$$f(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi v}} \left(1 + \frac{x^2}{v}\right)^{-\frac{v+1}{2}},$$

where $\Gamma(\cdot)$ is the gamma function.

Question B.1: Give a brief interpretation of the model.
Argue that (y_t, s_t) is a Markov chain.
Argue that the conditional density of (y_t, s_t) satisfies

$$f((y_t, s_t)|y_{t-1}, s_{t-1}) = f(y_t|s_t) f(s_t|s_{t-1}).$$

Question B.2: Suppose that the Markov chain (s_t) is irreducible and aperiodic. Show that

$$E[y_t] = 0$$

and

$$E[y_t^2] = \frac{1 - p_{22}}{2 - p_{11} - p_{22}} \left(\frac{v_1}{v_1 - 2}\right) + \frac{1 - p_{11}}{2 - p_{11} - p_{22}} \left(\frac{v_2}{v_2 - 2}\right).$$

Question B.3: Suppose that we want to estimate the model parameters $\theta = (v_1, v_2)'$. Based on a sample $(y_0, y_1, \ldots, y_T)$ the log-likelihood function (conditional on y_0) is given by

$$L_T(\theta) = \sum_{t=1}^{T} \log f(y_t|y_{t-1}, \ldots, y_0).$$

Show that

$$f(y_t|y_{t-1}, \ldots, y_0) = \sum_{i=1}^{2} f(y_t|s_t = i) P(s_t = i|y_{t-1}, \ldots, y_0),$$

with

$$f(y_t|s_t = i) = \frac{\Gamma\left(\frac{v_i+1}{2}\right)}{\Gamma\left(\frac{v_i}{2}\right)\sqrt{\pi v_i}} \left(1 + \frac{y_t^2}{v_i}\right)^{-\frac{v_i+1}{2}}.$$

Explain briefly how you would compute $P(s_t = i|y_{t-1}, \ldots, y_0)$.

Question B.4: Let $\tau_{\text{risk}} > 0$ denote some constant risk threshold, and define the (conditional) probability of a loss exceeding τ_{risk} at time $T+1$,

$$\varsigma_{T+1}(\tau_{\text{risk}}) \equiv P(-y_{T+1} \geq \tau_{\text{risk}}|y_T, y_{T-1}, \ldots, y_0).$$

Let $\mathcal{T}_i : \mathbb{R} \to [0, 1]$ denote the cdf of a Student's t-distribution with v_i degrees of freedom, $i = 1, 2$.

Show that

$$\varsigma_{T+1}(\tau_{\text{risk}}) = \sum_{i=1}^{2} \mathcal{T}_i(-\tau_{\text{risk}}) P(s_{T+1} = i|y_T, y_{T-1}, \ldots, y_0).$$

Discuss briefly how you would estimate $\varsigma_{T+1}(\tau_{\text{risk}})$ based on a sample $(y_0, \ldots, y_T)$.