Written Exam at the Department of Economics winter 2020-21

Financial Econometrics A

Final Exam

19 January, 2021

(3-hour open book exam)

Answers only in English.

The paper must be uploaded as one PDF document. The PDF document must be named with exam number only (e.g. '127.pdf') and uploaded to Digital Exam.

This exam question consists of 5 pages in total (including this front page).

This exam has been changed from a written Peter Bangsvej exam to a take-home exam with helping aids. Please read the following text carefully in order to avoid exam cheating.

Be careful not to cheat at exams!
You cheat at an exam, if you during the exam:

- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text. This also applies to text from old grading instructions.
- Make your exam answers available for other students to use during the exam
- Communicate with or otherwise receive help from other people
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Use parts of a paper/exam answer that you have submitted before and received a passed grade for without making use of source referencing (self plagiarism)

You can read more about the rules on exam cheating on the study information pages in KUnet and in the common part of the curriculum section 4.12.

Exam cheating is always sanctioned with a warning and dispelling from the exam. In most cases, the student is also expelled from the university for one semester.

Please note there is a total of **8** questions that you should provide answers to. That is, **4** questions under *Question A*, and **4** under *Question B*.

Question A:

Consider the ARCH model given by

$$x_t = \sigma_t z_t,$$
$$\sigma_t^2(\alpha) = \omega + \alpha x_{t-1}^2,$$

with z_t $t_v(0,1)$ distributed, x_0 fixed and $t = 1, 2, ..., T$. With $v > 2$ and $\omega > 0$ fixed, the log-likelihood function in terms of $\alpha \geq 0$ is given by

$$\ell_T^{t_v}(\alpha) = -\tfrac{1}{2}\sum_{t=1}^{T}\left(\log\sigma_t^2(\alpha) + (v+1)\log\left(1 + \frac{x_t^2}{\sigma_t^2(\alpha)(v-2)}\right)\right).$$

As usual with α set to the true value $\alpha = \alpha_0$, we set $\sigma_t^2(\alpha_0) = \sigma_t^2$.

Question A.1: We wish to find a value for α, α_a say such that x_t is stationary, weakly mixing and $E|x_t| < \infty$ for $\alpha \in [0, \alpha_a)$ and $v = 4$. To do so apply the drift function

$$\delta(x) = 1 + |x|,$$

and use $E|z_t| = \sqrt{2}/2 \simeq 0.7$ to find α_a. It follows that $\alpha_a > 2$. Discuss this by comparing with the ARCH(1) model where z_t are iidN$(0,1)$.

Hint: Recall the inequality that $|a+b|^\delta < |a|^\delta + |b|^\delta$ for $a, b \in \mathbb{R}$ and $\delta \in (0,1)$.

Question A.2: It follows that

$$S_T = \partial \ell_T^{t_v}(\alpha)/\partial\alpha\big|_{\alpha=\alpha_0, v=4} = -\tfrac{1}{2}\sum_{t=1}^{T}\frac{x_{t-1}^2}{\sigma_t^2}\eta_t,$$

with η_t iid with $E\eta_t = 0$, and

$$\eta_t = \left(1 - 5\frac{z_t^2/2}{1+z_t^2/2}\right).$$

Argue that $E\left(\frac{z_t^2/2}{1+z_t^2/2}\right)^2 \leq 1$, and hence that $\sigma_\eta^2 = E\eta_t^2 < \infty$.

Argue that $\gamma = E\left(x_{t-1}^2/\sigma_t^2\right)^2 \leq 1/\alpha_0^2$ for any $\alpha_0 > 0$.

Question A.3: Use Question A.2 (and A.1) to show that

$$T^{-1}\sum_{t=1}^{T}\left(\frac{x_{t-1}^2}{\sigma_t^2}\right)^2 \overset{P}{\to} \gamma.$$

Next, show that

$$T^{-1/2}S_T = -T^{-1/2}\tfrac{1}{2}\sum_{t=1}^{T}\frac{x_{t-1}^2}{\sigma_t^2}\eta_t \overset{D}{\to} N\left(0, \gamma\sigma_\eta^2/4\right)$$

for any $\alpha_0 \in (0, \alpha_a)$.

Similarly one can show that if one relaxes that v is known,

$$S_T^v = \partial \ell_T^{t_v}(\alpha, v)/\partial v\big|_{\alpha=\alpha_0, v=v_0}$$

is asymptotically Gaussian distributed.

Discuss implications of these results and in particular why $\alpha_0 > 0$ is an important assumption.

Question A.4: With a sample of $T = 1000$ observations, it follows that the MLE of v and α, and corresponding LR test statistics for the hypotheses $H_v : v = 4$ and $H_\alpha : \alpha = 0$ are given by:

MLE	Value	Hypothesis	LR statistic
$\hat{v}$	4.7	$H_v : v = 4$	$LR(v = 4) = 2.2$
$\hat{\alpha}$	0.06	$H_\alpha : \alpha = 0$	$LR(\alpha = 0) = 3.1$

Discuss if you would reject H_v and/or H_α. Be precise about which asymptotic distribution(s) and quantiles you are applying.

Question B:

In order to introduce a stochastic jump in log-returns x_t, consider the "Jump-ARCH" model for x_t as given by

$$x_t = \varepsilon_t + J_t, \quad t = 1, 2, ..., T.$$

With the initial value x_0 fixed, ε_t is an "ARCH" component as given by

$$\varepsilon_t = \sigma_t z_t,$$

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2, \quad \omega > 0, \quad \alpha \geq 0,$$

with z_t iidN$(0, 1)$. Note that it is x_t lagged that enters the σ_t^2 and not as for a standard ARCH model, ε_t lagged.

Next the J_t is a "jump" component which is given by a sum of a random number s_t of random $\eta_{t,i}$ variables which are iidN$(0, \gamma)$, with $\gamma > 0$. That is,

$$J_t = \eta_{t,1} + ... + \eta_{t,s_t} = \sum_{i=1}^{s_t} \eta_{t,i},$$

with s_t stochastic and taking values in $1, 2$ or 3. More specifically, we consider here the case where s_t is given by a 3-state Markov chain with constant transition probabilities $p_{ij} = P(s_t = j | s_{t-1} = i) \in [0, 1]$ for $i, j = 1, 2, 3$, such that $\sum_{j=1}^{3} p_{ij} = 1$ for $i = 1, 2, 3$.

Throughout, we assume that the processes $(z_t)_{t=1,2,...}$ and $(\eta_{t,i})_{t=1,2,....}$ are independent for every $i = 1, 2, 3$, and that the processes $(\eta_{t,i})$ and $(\eta_{t,j})$ are independent for $i \neq j$. Lastly, we also assume that the Markov chain (s_t) is independent of (z_t) and $(\eta_{t,i})$ for every $i = 1, 2, 3$.

Question B.1: State conditions on the transition probabilities $(p_{ij})_{i,j=1,2,3}$ which implies that s_t is weakly mixing.
Note: You do not have to provide any derivations.

Question B.2: State the conditional density of x_t given J_t and $(x_{t-1}, \ldots, x_0)$. That is, give an expression for

$$f(x_t | J_t, x_{t-1}, \ldots, x_0),$$

for $t \geq 1$. Use this to give an interpretation of the model for the log-returns x_t conditional on J_t and past x_t's.

Question B.3: Here we consider the model without conditioning on the "Jumps" J_t but conditional on the value of s_t (and lagged x_t's).

Argue that J_t conditional on s_t is $N\left(0, \sum_{j=1}^{s_t} \gamma\right)$ distributed.

Next, use this to show that for $t \geq 1$ the conditional density of x_t given $s_t = i$ and $(x_{t-1}, \ldots, x_0)$ is given by

$$f(x_t|s_t = i, x_{t-1}, \ldots, x_0) = \frac{1}{\sqrt{2\pi(\sigma_t^2 + i\gamma)}} \exp\left(-\frac{x_t^2}{2(\sigma_t^2 + i\gamma)}\right), \quad i = 1, 2, 3.$$

Give an interpretation of the model in this case.

Question B.4: Let $\theta = (\omega, \alpha, \gamma, p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32})'$ denote the model parameters. The log-likelihood function is given by

$$L_T(\theta) = \sum_{t=1}^{T} \log f_\theta(x_t|x_{t-1}, \ldots, x_0).$$

Explain how you would estimate θ. In particular, explain how the log-likelihood function can be evaluated.