Written Exam at the Department of Economics winter 2018-19

Financial Econometrics A

Final Exam

February 15, 2019

(3-hour closed book exam)

Answers only in English.

This exam question consists of 5 pages in total

NB: If you fall ill during an examination at Peter Bangs Vej, you must contact an invigilator who will show you how to register and submit a blank exam paper. Then you leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Be careful not to cheat at exams!

- You cheat at an exam, if during the exam, you:
- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Please note there is a total of **9 questions** that you should provide answers to. That is, 5 questions under Question A, and 4 under Question B.

Question A:

Consider the ARCH(1) model,

$$x_t = \sigma_t z_t, \tag{A.1}$$

with $z_t \sim i.i.d.N(0,1)$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2, \quad \omega > 0, \;\; 1 > \alpha \geq 0.$$

Question A.1: Suppose that x_t is stationary with $E(x_t^2) < \infty$, and define $\gamma := E(x_t^2)$. Find an expression for γ in terms of ω and α, and show that σ_t^2 can be re-written as

$$\sigma_t^2 = \gamma(1-\alpha) + \alpha x_{t-1}^2. \tag{A.2}$$

Question A.2: Show that x_t is stationary and weakly mixing with $E(x_t^4) < \infty$ if $\alpha < 1/\sqrt{3}$. (Hint: Recall that $E[z_t^4] = 3$.)

Question A.3: In the following, we define the vector of parameters in the model as $\theta = (\gamma, \alpha)'$. The log-likelihood contribution at time t, $l_t(\theta)$, in terms of (A.1) and (A.2) is (up to a scaling constant),

$$l_t(\theta) = -\log(\sigma_t^2(\theta)) - \frac{x_t^2}{\sigma_t^2(\theta)}, \quad \sigma_t^2(\theta) = \gamma(1-\alpha) + \alpha x_{t-1}^2.$$

Show that the score in the direction of α evaluated at the true parameters $\theta_0 = (\gamma_0, \alpha_0)$ is given by

$$\left.\frac{\partial l_t(\theta)}{\partial \alpha}\right|_{\theta=\theta_0} = \frac{\left(x_{t-1}^2 - \gamma_0\right)}{\gamma_0 + \alpha_0\left(x_{t-1}^2 - \gamma_0\right)}\left(z_t^2 - 1\right).$$

Question A.4: Assume that $0 < \alpha_0 < 1$. Show that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left.\frac{\partial l_t(\theta)}{\partial \alpha}\right|_{\theta=\theta_0} \xrightarrow{D} N(0,\xi), \quad \text{as } T \to \infty. \tag{A.3}$$

Explain briefly how (A.3) can be used.

Question A.5: Suppose that one seeks to investigate whether the level, γ, of the volatility is beyond some given value, γ_0, i.e. $\gamma > \gamma_0$. This can be done by testing the null hypothesis

$$H_0 : \gamma = \gamma_0,$$

against the alternative $H_A : \gamma > \gamma_0$. Explain how you would test H_0 based on the maximum likelihood estimator for $\theta = (\gamma, \alpha)'$. Be specific about which conditions are needed.

Question B:

Consider the log-returns of a portfolio y_t given in Figure B.1 with $t = 1, 2, ..., T = 1000$.

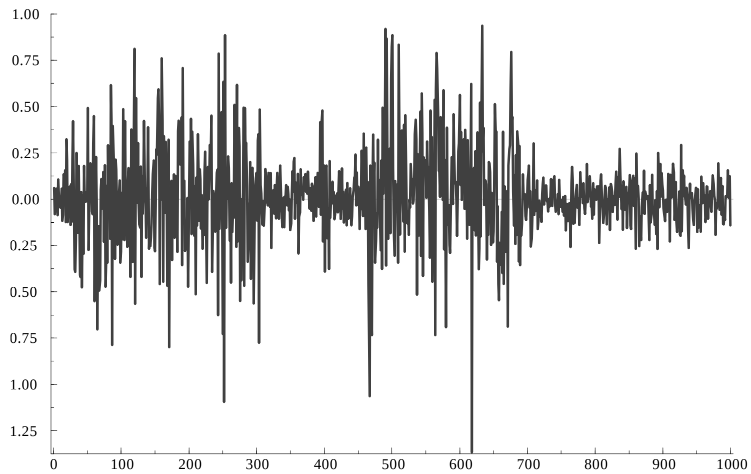

Figure B.1: Portfolio returns, y_t

Question B.1: Estimation with a 2-state Markov switching stochastic volatility model, gave the following output in the usual notation in terms of the transition matrix $P = (p_{ij})_{i,j=1,2}$ and smoothed standardized residuals, $\hat{z}_t^* = y_t/\hat{E}[\sigma_t|y_1, ..., y_T]$:

$\hat{P}$, QMLE of P:	$\hat{p}_{11} = 0.97$ $\hat{p}_{22} = 0.99$
	p-values for tests based on $\hat{z}_t^*$:
Normality test:	0.06
LM-test for no ARCH:	0.10

What would you conclude on the basis of the output and the graph?

Question B.2: In order to compute the Value-at-Risk (VaR) of the portfolio, the following ARCH-type model was proposed:

$$y_t = \sigma_{s_t,t} z_t \tag{B.1}$$

Here the process (z_t) is $i.i.d.N(0, 1)$ and independent of the $i.i.d.$ process (s_t), $s_t \in \{1, 2\}$, where $p = P(s_t = 1) = 1 - P(s_t = 2)$. Moreover,

$$\sigma_{1,t}^2 = \omega + \alpha y_{t-1}^2 \quad \text{and} \quad \sigma_{2,t}^2 = \gamma. \tag{B.2}$$

Thus the parameters of the model are $\theta = (\omega, \alpha, \gamma, p)$ with $\omega, \gamma > 0$, $\alpha \geq 0$ and $p \in [0, 1]$.

Provide a brief interpretation of the model.

Suppose, first that $(s_1, ..., s_T)$ is observed. Then, based on $(y_0, y_1, ..., y_T, s_1, ..., s_T)$, the log-likelihood function, $L_T(\theta)$, conditional on the initial value y_0 is given by

$$L_T(\theta) = \log f_\theta(y_1, ..., y_T, s_1, ..., s_T | y_0).$$

Argue that

$$f_\theta(y_1, ..., y_T, s_1, ..., s_T | y_0) = \prod_{t=1}^{T} f_\theta(y_t | s_t, y_{t-1}) p_\theta(s_t),$$

and provide expressions for $f_\theta(y_t | s_t, y_{t-1})$ and $p_\theta(s_t)$.

Question B.3: Next, suppose that s_t is unobserved. Then one may instead of $L_T(\theta)$ consider the function

$$\begin{aligned} L_T^\dagger(\theta) = & \sum_{t=1}^{T} p_t^\dagger \left\{ \log f_\theta(y_t | y_{t-1}, s_t = 1) + \log(p) \right\} \\ & + \sum_{t=1}^{T} (1 - p_t^\dagger) \left\{ \log f_\theta(y_t | y_{t-1}, s_t = 2) + \log(1 - p) \right\}, \end{aligned}$$

where

$$p_t^\dagger = P_{\theta^\dagger}[s_t = 1 | y_0, y_1, ..., y_T]$$

for some fixed $\theta^\dagger$.

Discuss briefly estimation of θ based on $L_T^\dagger(\theta)$.

Question B.4: The 5% level Value at Risk, $\text{VaR}_{T+1}^{5\%}$, satisfies:

$$P\left(y_{T+1} \leq -\text{VaR}_{T+1}^{5\%} \mid y_0, y_1, ..., y_T \right) = 5\%.$$

Suppose that p is known and fixed with $p = 1$. Then

$$\text{VaR}_{T+1}^{5\%} = -\sigma_{1,T+1} \Phi^{-1}(0.05),$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution.

Explain briefly how you would compute an estimate of $\text{VaR}_{T+1}^{5\%}$.

Explain briefly how you would compute an estimate of $\text{VaR}_{T+1}^{5\%}$ if instead $p = 1/2$.