Written Exam at the Department of Economics winter 2018-19

Financial Econometrics A

Final Exam

January 16, 2019

(3-hour closed book exam)

Answers only in English.

This exam question consists of 6 pages in total

NB: If you fall ill during an examination at Peter Bangs Vej, you must contact an invigilator who will show you how to register and submit a blank exam paper. Then you leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Be careful not to cheat at exams!

- You cheat at an exam, if during the exam, you:
- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Please note there is a total of **9 questions** that you should provide answers to. That is, 5 questions under Question A, and 4 under Question B.

Question A:

Consider the time series model given by,

$$x_t = \sigma_t z_t, \quad t = 1, 2, ... \tag{A.1}$$

with $z_t \sim i.i.d.N(0,1)$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta g(y_{t-1}), \tag{A.2}$$

with $\omega > 0$, $\alpha, \beta \geq 0$. Here y_t is some exogenous covariate, as for example the realized volatility, and $g(\cdot)$ is a *continuous* function satisfying $g(y_t) \geq 0$ for all t. The initial x_0 and y_0 are taken as given.

Question A.1: Suppose that $\beta = 0$. State a condition for x_t to be weakly mixing and such that $Ex_t^2 < \infty$. You do not have to provide any derivations.

Question A.2: Suppose that $\beta > 0$. Assume that y_t is $i.i.d.N\left(0, \sigma_y^2\right)$, and that the processes (z_t) and (y_t) are independent.
With $v_t = (x_t, y_t)'$ it holds that the density of v_t conditional on $(v_0, v_1, ..., v_{t-1})$ is given by

$$f\left(v_t | v_{t-1}, ..., v_0\right) = f(x_t | v_{t-1}) f(y_t | v_{t-1}), \quad t \geq 1.$$

Argue that v_t is a Markov chain for which the transition density $f\left(\cdot|\cdot\right)$ is such that the drift criterion can be applied.

Next, suppose that $g(y_t) = y_t^2$. With $v = (x, y)'$ let $\|v\|^2 = v'v = x^2 + y^2$. With drift function $\delta\left(v_t\right) = 1 + \|v_t\|^2$, show that for some constant c,

$$E\left(\delta\left(v_t\right) | v_{t-1} = v\right) \leq c + \max\left(\alpha, \beta\right)\left(x^2 + y^2\right).$$

Conclude that if $\max\left(\alpha, \beta\right) < 1$, then v_t is weakly mixing with $E[x_t^2] + E[y_t^2] < \infty$.

Question A.3: Suppose that y_t is not necessarily $i.i.d.N(0, \sigma_y^2)$ and that $g(y_t)$ is not necessarily equal to y_t^2.
Let $\theta = (\omega, \alpha, \beta)'$. With $L_T\left(\theta\right)$ the log-likelihood function for the model, it holds that the score for β is given by,

$$S\left(\theta\right) = \partial \log L_T\left(\theta\right) / \partial \beta = \sum_{t=1}^{T} \frac{1}{2}\left(\frac{x_t^2}{\sigma_t^2(\theta)} - 1\right)\frac{g(y_{t-1})}{\sigma_t^2(\theta)},$$

with

$$\sigma_t^2(\theta) = \omega + \alpha x_{t-1}^2 + \beta g(y_{t-1}).$$

Suppose that $(x_t, g(y_t))'$ is weakly mixing and that the true parameter values $\theta_0 = (\omega_0, \alpha_0, \beta_0)'$ satisfy $\omega_0 > 0$, $\alpha_0 < 1$, and $0 < \beta_L \leq \beta_0 < 1$. Show that as $T \to \infty$

$$\frac{1}{\sqrt{T}} S(\theta_0) \xrightarrow{d} N\left(0, \frac{\nu}{2}\right), \tag{A.3}$$

where

$$\nu = E\left[\left(\frac{g(y_{t-1})}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 g(y_{t-1})}\right)^2\right] \leq 1/\beta_L^2 < \infty.$$

Question A.4: Explain briefly what (A.3) can be used for.

Question A.5: Suppose now that y_t is the square-root of the Realized Volatility based on 10-minutes intraday log-returns on the S&P500 Index, denoted $RV_t^{1/2}$. A plot of $RV_t^{1/2}$ and its sample autocorrelation function is given in Figure A.1.

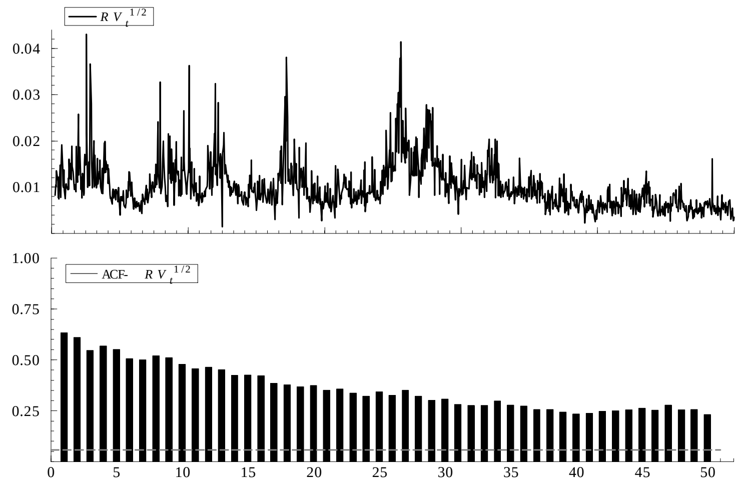

Figure A.1

Maximum likelihood estimation of the parameters of the model (A.1)-(A.2) with $g(RV_t^{1/2}) = RV_t^{1/2}$ gave the following output:

Output: MLE of ARCH with RV	
$\hat{\alpha} = 0.07$	std.deviation$(\hat{\alpha}) = 0.012$
$\hat{\beta} = 0.21$	std.deviation$(\hat{\beta}) = 0.121$

What would you conclude in terms of the importance of Realized Volatility?

Question B:

Question B.1: As part of a discussion of "bubbles" in financial markets, consider the asset log-price series y_t in Figure B.1 with $t = 1, 2, ..., 1620$.

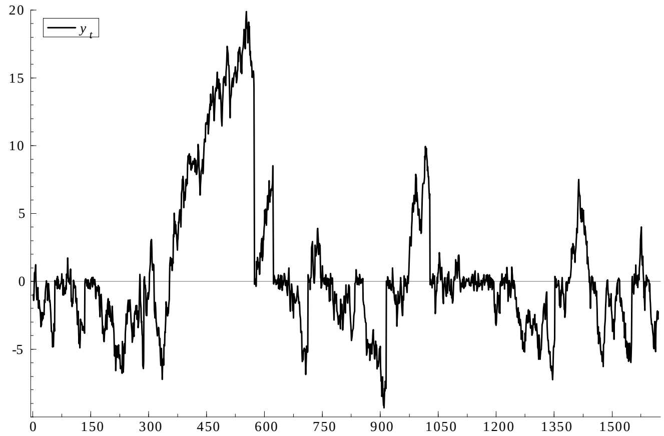

Figure B.1

For estimation, the following 2-state Markov switching model was applied:

$$y_t = \rho_{s_t} y_{t-1} + \sigma_{s_t} z_t, \quad t = 2, ..., T, \quad T = 1620. \tag{B.1}$$

Here z_t is $i.i.d.N(0,1)$ distributed and y_1 is fixed. Moreover, $s_t \in \{1, 2\}$ is an unobserved state variable governed by the transition matrix $P = (p_{ij})_{i,j=1,2}$ with $p_{ij} = P(s_t = j | s_{t-1} = i)$. The processes (z_t) and (s_t) are independent. It holds that

$$\rho_{s_t} = \rho 1 (s_t = 1) \quad \text{and} \quad \sigma^2_{s_t} = \sigma^2_1 1 (s_t = 1) + \sigma^2_2 1 (s_t = 2). \tag{B.2}$$

Gaussian likelihood estimation gave the following output, with misspecification tests in terms of smoothed standardized residuals $\hat{z}^*_t$:

MLE of P:	$\hat{p}_{11} = 0.98 \quad \hat{p}_{21} = 0.07$
MLE of ρ :	$\hat{\rho} = 0.99$
MLE of σ^2_1 and σ^2_2	$\hat{\sigma}^2_1 = 0.71$ and $\hat{\sigma}^2_2 = 0.30$
	p-values:
Test for Normality of $\hat{z}^*_t$:	0.12
LM-test for no ARCH in $\hat{z}^*_t$:	0.10
LR-test of $\rho = 1$:	0.81

Interpret the model. What would you conclude on the basis of the output and Figure B.1?

Question B.2: In order to obtain an estimate of $\theta = (p_{11}, p_{22}, \rho, \sigma_1^2, \sigma_2^2)$, the function $M(\theta)$ given by

$$M(\theta) = \sum_{i,j=1}^{2} \log p_{ij} \sum_{t=2}^{1620} p_t^*(i,j) + \sum_{j=1}^{2} \sum_{t=2}^{1620} p_t^*(j) \log f_\theta (y_t | y_{t-1}, s_t = j), \quad \text{(B.3)}$$

can be used. Provide an expression for $f_\theta (y_t | y_{t-1}, s_t = 1)$.

Question B.3: Explain how you would use $M(\theta)$ from (B.3) in order to find and estimate, $\hat{\theta}$, of θ.

Comment briefly on what Figure B.2 shows in relation to finding $\hat{\theta}$.

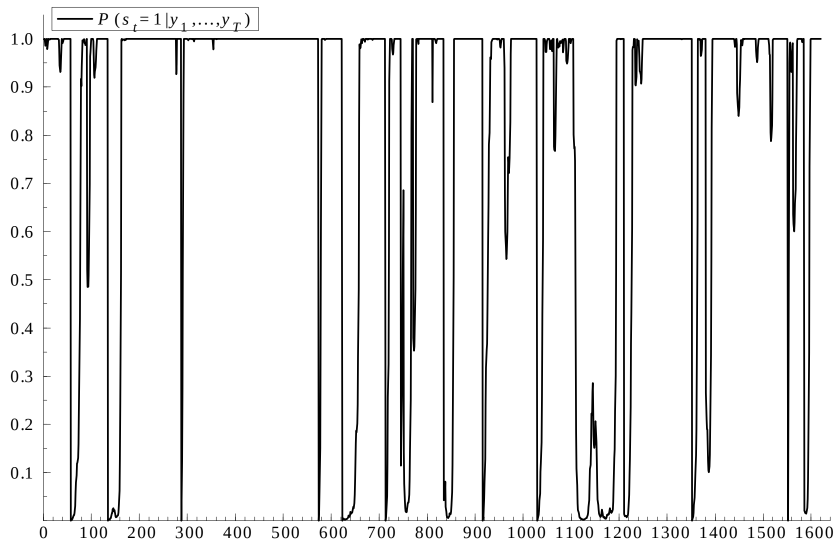

Figure B.2: $P(s_t = 1 | y_1, ..., y_T)$

Question B.4: Now assume that at time T, $s_T = 1$. In order to forecast if one will enter state 2 at $T+2$, derive

$$P(s_{T+2} = 2 | s_T = 1),$$

and provide an estimate of this given the estimation output in Question B.1.