Written Exam at the Department of Economics winter 2016-17

# Financial Econometrics A

Final Exam

Date: January 6th, 2017

(3-hour closed book exam)

Please note that the language used in your exam paper must correspond to the language for which you registered during exam registration.

**This exam question consists of 5 pages in total**

*NB: If you fall ill during the actual examination at Peter Bangsvej, you must contact an invigilator in order to be registered as having fallen ill. Then you submit a blank exam paper and leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.*

Please note there are a total of **9** questions that you should provide answers to. That is, **4** questions under *Question A*, and **5** under *Question B*.

# Question A:

Consider the following log-linear Realized GARCH model given by

$$x_t = \sigma_t z_t, \tag{A.1}$$

with $z_t \sim i.i.d.N(0,1)$, and

$$\log(\sigma_t^2) = 1 + \alpha \log(y_{t-1}), \tag{A.2}$$
$$\log(y_t) = \gamma + \phi \log(\sigma_t^2) + u_t, \tag{A.3}$$

with $u_t \sim i.i.d.N(0,1)$ and $\alpha, \gamma, \phi \in \mathbb{R}$. It is assumed that the processes $(z_t)$ and $(u_t)$ are independent. Here $y_t$ is some *observed* positive exogenous covariate as for example the realized volatility.

**Question A.1:** Use the drift criterion to show that $\log(y_t)$ is weakly mixing with $E[(\log(y_t))^2] < \infty$, if $|\alpha\phi| < 1$.
Given that $\log(y_t)$ is weakly mixing we do also have that the joint process $(x_t, \log(y_t))$ is weakly mixing.

**Question A.2:** Let $\theta = (\alpha, \gamma, \phi)$ denote the model parameters. Given a sample $(x_t, \log(y_t))$, $t = 0, 1, ..., T$, the joint log-likelihood is (up to a constant term and a scaling factor)

$$L_T(\theta) = \sum_{t=1}^{T} l_t(\theta),$$

$$l_t(\theta) = -\log(\sigma_t^2(\theta)) - \frac{x_t^2}{\sigma_t^2(\theta)} - \left[\log(y_t) - \gamma - \phi\log(\sigma_t^2(\theta))\right]^2,$$

where $\log(\sigma_t^2(\theta)) = 1 + \alpha\log(y_{t-1})$.
Show that

$$\frac{\partial l_t(\theta)}{\partial \alpha} = \left\{\frac{x_t^2}{\sigma_t^2(\theta)} - 1 + 2\phi\left[\log(y_t) - \gamma - \phi\log(\sigma_t^2(\theta))\right]\right\}\log(y_{t-1}).$$

*Hint*: You may want to use that

$$\frac{\partial l_t(\theta)}{\partial \alpha} = \frac{\partial l_t(\theta)}{\partial \log(\sigma_t^2(\theta))}\frac{\partial \log(\sigma_t^2(\theta))}{\partial \alpha}.$$

**Question A.3:** Let $\theta_0 = (\alpha_0, \gamma_0, \phi_0)$ denote the vector of true parameter values. Define $S_T(\theta) = \partial L_T(\theta)/\partial\alpha$.
Assume that $(x_t, \log(y_t))$ is weakly mixing and satisfies the drift criterion such that $E[(\log(y_{t-1}))^2] < \infty$. Show that

$$\frac{1}{\sqrt{T}} S_T(\theta_0) \xrightarrow{d} N(0, v), \qquad \text{(A.4)}$$

where $v = (2 + 4\phi_0^2)E[(\log(y_{t-1}))^2]$.
Explain briefly what the property (A.4) can be used for.

*Hint*: Use that $\log(y_t) - \gamma_0 - \phi_0\log(\sigma_t^2(\theta_0)) = u_t$. Moreover, you may want to recall that $E[z_t^4] = 3$.

**Question A.4:** For the model (A.1)-(A.3), the one-period VaR at risk level $\kappa$, $\mathrm{VaR}_{T,1}^{\kappa}$, is defined as

$$P_T(x_{T+1} < -\mathrm{VaR}_{T,1}^{\kappa}) = \kappa, \quad \kappa \in (0, 1),$$

where $P_T(\cdot)$ denotes the conditional distribution of $x_{T+1}$. It can be shown (but do not do so) that

$$\mathrm{VaR}_{T,1}^{\kappa} = -\sigma_{T+1}\Phi^{-1}(\kappa),$$

where $\Phi^{-1}(\cdot)$ denotes the inverse cdf of the standard normal distribution.
Explain briefly how you would compute an estimate of $\mathrm{VaR}_{T,1}^{\kappa}$.

# Question B:

Consider the following switching model given by

$$x_t = \mu 1_{(s_t=1)} + \varepsilon_t, \tag{B.1}$$

where $\mu$ is an $\mathbb{R}$-valued constant and $s_t$ can take value 1 or 2. Moreover, $\varepsilon_t \sim i.i.d.N(0, \sigma^2)$, and we assume that the processes $(s_t)$ and $(\varepsilon_t)$ are independent. Suppose that $s_t$ is a two-state Markov chain with transition probabilities $P(s_t = j|s_{t-1} = i) = p_{ij}$, $i, j = 1, 2$.
Note that $1_{(s_t=1)} = 1$ if $s_t = 1$ and $1_{(s_t=1)} = 0$ if $s_t = 2$.

**Question B.1:** Suppose that $\mu = 0$. Explain if $x_t$ is weakly mixing.
What should hold for $p_{11}$ and $p_{22}$ for $s_t$ to be weakly mixing?

**Question B.2:** Next, assume that $s_t$ is *observed*. Moreover, suppose that the transition probabilities satisfy $p_{11} = (1 - p_{22}) = p \in (0, 1)$ such that $s_t$ is and *i.i.d.* process with $P(s_t = 1) = p$ and $P(s_t = 2) = 1 - p$.
Show that for $t \geq 1$, the joint conditional density of $(x_t, s_t)$ is

$$\begin{aligned} f(x_t, s_t|x_{t-1}, s_{t-1}, ..., x_0, s_0) = & \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_t - \mu)^2}{2\sigma^2}\right) p\right]^{1_{(s_t=1)}} \\ & \times \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{{x_t}^2}{2\sigma^2}\right) (1 - p)\right]^{1_{(s_t=2)}}. \end{aligned}$$

**Question B.3:** Maintaining the assumptions from Question B.2, let $\theta = (\mu, \sigma^2, p)$ denote the model parameters. The log-likelihood function is

$$\begin{aligned} L_T(\theta) = & \sum_{t=1}^{T} \left\{\log(p) - \frac{1}{2}\log(2\pi\sigma^2) - \frac{(x_t - \mu)^2}{2\sigma^2}\right\} 1_{(s_t=1)} \\ & + \sum_{t=1}^{T} \left\{\log(1 - p) - \frac{1}{2}\log(2\pi\sigma^2) - \frac{{x_t}^2}{2\sigma^2}\right\} 1_{(s_t=2)}. \end{aligned}$$

Let $\hat{\mu}$ denote the maximum likelihood estimator for $\mu$.
Show that

$$\hat{\mu} = \frac{\sum_{t=1}^{T} x_t 1_{(s_t=1)}}{\sum_{t=1}^{T} 1_{(s_t=1)}}.$$

Moreover, let $\hat{p}$ denote the maximum likelihood estimator for $p$. Derive $\hat{p}$ and argue that $\hat{p} \xrightarrow{P} p$ as $T \to \infty$.

**Question B.4:** Suppose that the process $(s_t)$ is *unobserved*, but does still satisfy the *i.i.d.* assumption, i.e. $p_{11} = (1 - p_{22}) = p \in (0, 1)$. Then the estimators derived in Question B.3 are infeasible. Instead we may introduce

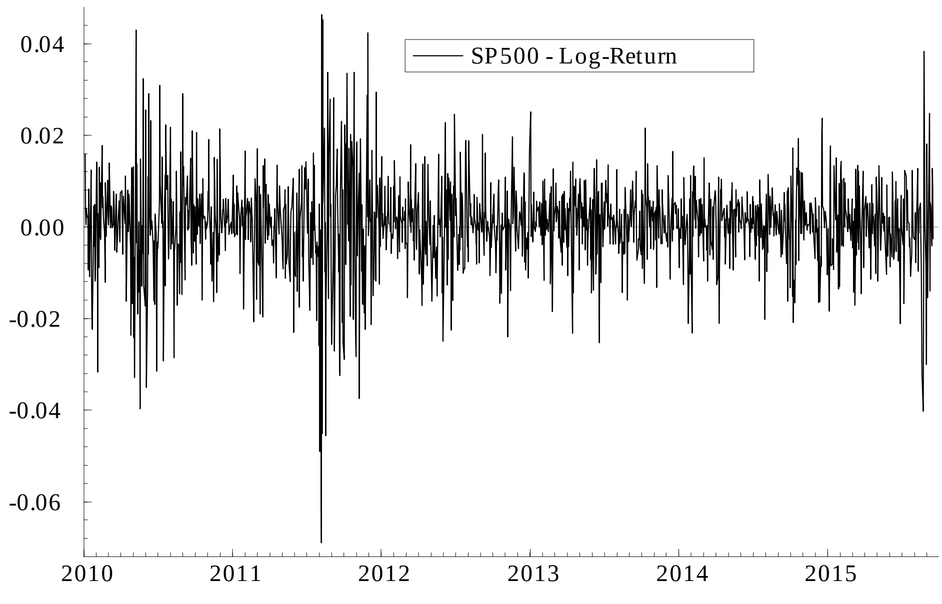
$$\tilde{L}_T(\theta) = E[L_T(\theta)|x_1, ...x_T].$$

It holds that

$$\tilde{L}_T(\theta) = \sum_{t=!}^{T} \left\{ \log(p) - \frac{1}{2}\log(2\pi\sigma^2) - \frac{(x_t - \mu)^2}{2\sigma^2} \right\} P_t^{\star}(1)$$
$$+ \sum_{t=1}^{T} \left\{ \log(1-p) - \frac{1}{2}\log(2\pi\sigma^2) - \frac{{x_t}^2}{2\sigma^2} \right\} (1 - P_t^{\star}(1)),$$

where $P_t^{\star}(1) = P(s_t = 1|x_t)$.
Explain briefly the role of $\tilde{L}_T(\theta)$ for the estimation of $\theta$.

**Question B.5:** The following figure shows the daily log-returns of the S&P 500 index for the period January 4, 2010 to September 17, 2015.




Discuss briefly whether the swithcing model in (B.1) is adequate for modelling the main features of the log-returns. Would another type of Markov switching model be more suitable?