Written Exam for the B.Sc. in Economics. Winter 2011 - 2012

DYNAMIC MODELS

2nd year of study

Wednesday, January 18, 2012

(3 hours written exam. All usual aids allowed (i. e. books, notes etc.), but it is neither allowed bringing any electronic calculator nor using any other electronic equipment. Open book exam)

Please note that the language used in your exam must correspond to the language of the title for which your registered during exam registration. I. e. if you registered for the English title of the course, you must solve the English set in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title followed by "eksamen på dansk" in brackets, you must solve the Danish exam set in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

UNIVERSITY OF COPENHAGEN
DEPARTMENT OF ECONOMICS

2nd year of study. 2012 W-2DM ex

WRITTEN EXAM. DYNAMIC MODELS

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3 pages with 4 problems.

Duration: 3 hours.

It is allowed using textbooks, lecture notes, and personal notes. It is strictly prohibited using any electronic calculator or cas tool.

Problem 1. We consider the polynomial $P : \mathbf{C} \to \mathbf{C}$ that is given by

$$\forall\, z \in \mathbf{C} : P(z) = 2z^4 + 2z^3 + 7z^2 + 2z + 5.$$

Furthermore, we consider the differential equation

$$(*) \qquad \frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + \frac{7}{2}\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{5}{2}x = 0$$

and the differential equations

$$(**) \qquad \frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + \frac{7}{2}\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{5}{2}x = 27e^t$$

and

$$(***) \qquad \frac{d^5y}{dt^5} + \frac{d^4y}{dt^4} + \frac{7}{2}\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{5}{2}\frac{dy}{dt} = 0.$$

(1) Show that the complex numbers i and $-i$ are roots of the polynomial P, i. e. $P(i) = 0$ and $P(-i) = 0$.

(2) Solve the equation
$$P(z) = 0.$$

(3) Determine the general solution of the differential equation $(*)$.

(4) Show that the differential equation $(*)$ is not globally asymptotically stable.

(5) Determine the general solution of the differential equation $(**)$.

(6) Determine the general solution of the differential equation $(***)$.

Problem 2. Consider the vector space $\mathbf{R}^n$, where $n \in \mathbf{N}$ and $n \geq 3$. Also consider the set

$$S = \{x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n : x_1 > 0 \wedge x_2 > 0\}.$$

(1) Show that the set S is an open subset of $\mathbf{R}^n$.

(2) Find the closure $\overline{S}$ of the set S.

(3) Find the complement $\mathcal{C}S$ of the set S and find the boundary $\partial(\mathcal{C}S)$ of this set.

(4) Is the set $\mathcal{C}S$ closed?

Problem 3. We consider the vector valued function $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$\forall\,(x, y) \in \mathbf{R}^2 : f(x, y) = \begin{pmatrix} 2xy + e^y \\ e^x + 4y^2 \end{pmatrix}.$$

(1) Find the Jacobi matrix $Df(x, y)$ of the function f at any point $(x, y) \in \mathbf{R}^2$.

(2) Find the determinant $\det Df(x, y)$ and show that the Jacobi matrix $Df(0, 0)$ is non-singular.

(3) Prove that there exists a neighbourhood $U_{(0,0)}$ of the point $(0, 0)$ such that the Jacobi matrix $Df(x, y)$ is non-singular at any point $(x, y) \in U_{(0,0)}$.

(4) Find the inverse $(Df(0, 0))^{-1}$ of the non-singular Jacobi matrix $Df(0, 0)$.

(5) Solve the equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = f(0, 0) + Df(0, 0) \begin{pmatrix} x \\ y \end{pmatrix}$$

with respect to

$$\begin{pmatrix} x \\ y \end{pmatrix}.$$

(6) Show that the vector valued function $g : \mathbf{R}^2 \to \mathbf{R}^2$ given by the rule

$$\forall\,(x, y) \in \mathbf{R}^2 : g(x, y) = f(0, 0) + Df(0, 0) \begin{pmatrix} x \\ y \end{pmatrix}$$

has no fixed points.

Problem 4. We consider and the function $F : \mathbf{R}^3 \to \mathbf{R}$ given by the rule

$$\forall\,(t,x,y) \in \mathbf{R}^3 : F(t,x,y) = y^2 + (1+t^2)x.$$

Furthermore, we consider the functional

$$I(x) = \int_0^1 \Big(\Big(\frac{dx}{dt}\Big)^2 + (1+t^2)x \Big)\, dt.$$

(1) Show that for every $t \in \mathbf{R}$ the function $F = F(t,x,y)$ is convex in $(x,y) \in \mathbf{R}^2$.

(2) Solve the variational problem: Determine the minimum function $x^* = x^*(t)$ of the functional $I(x)$ subject to the conditions

$$x^*(0) = 3 \ \text{ and } \ x^*(1) = \frac{1}{24}\,.$$