Written Exam for the B.Sc. in Economics. Winter 2011 - 2012

DYNAMIC MODELS

2nd year of study

Thursday, February 23, 2012

(3 hours written exam. All usual aids allowed (i. e. books, notes etc.), but it is neither allowed bringing any electronic calculator nor using any other electronic equipment. Open book exam)

Please note that the language used in your exam must correspond to the language of the title for which your registered during exam registration. I. e. if you registered for the English title of the course, you must solve the English set in English. Likewise, if you registered for the Danish title of the course or if you registered for the English title followed by "eksamen på dansk" in brackets, you must solve the Danish exam set in Danish.

If you are in doubt about which title you registered for, please see the print of your exam registration from the students' self-service system.

UNIVERSITY OF COPENHAGEN
DEPARTMENT OF ECONOMICS

2nd year of study. 2012 W-2MA rx

WRITTEN EXAM. DYNAMIC MODELS

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3 pages with 4 problems.

Duration: 3 hours.

It is allowed using textbooks, lecture notes, and personal notes. It is strictly prohibited bringing any electronic calculator or using any cas tool.

Problem 1. We consider the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

and the vectorial differential equations

$$(\Delta) \qquad \frac{d\mathbf{z}}{dt} = A\mathbf{z}$$

and

$$(\Xi) \qquad \frac{d\mathbf{z}}{dt} = A\mathbf{z} + \begin{pmatrix} 4 \\ -4 \\ 8 \end{pmatrix}.$$

(1) Show that the matrix A is non-singular, i. e. A is invertible.

(2) Determine the inverse matrix A^{-1} of A.

(3) Find the eigenvalues of the matrix A.

(4) Find the eigenspaces of the matrix A.

(5) Determine the general solution of the vectorial differential equation (Δ).

(6) Determine the general solution of the vectorial differential equation (Ξ).

(7) For every $v \in \mathbf{R}$ we consider the 3×3 matrix

$$B(v) = \begin{pmatrix} v & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & v \end{pmatrix}$$

and the vectorial differential equation

$$(\S) \qquad \frac{d\mathbf{z}}{dt} = B(v)\mathbf{z}.$$

Show that the vectorial differential equation is not globally asymptotically stable for any value of $v \in \mathbf{R}$.

Problem 2. For any $r \geq 1$ we consider the set

$$K(r) = \{z \in \mathbf{C} : \frac{1}{r} \leq |z| \leq r\}.$$

(1) Show that, for any $r \geq 1$, the set $K(r)$ is compact.

(2) Find, for any $r \geq 1$, the interior $(K(r))^o$ of the set $K(r)$.

(3) Determine the sets

$$K = \bigcap_{r>1} K(r) \text{ and } K_\infty = \bigcup_{r>1} K(r).$$

(4) Show that the set K_∞ is open.

(5) Let (ζ_k) be a sequence of points such that

$$\forall\, k \in \mathbf{N} : \zeta_k \in K\Big(1 + \frac{1}{k}\Big).$$

Show that the sequence (ζ_k) has a convergent subsequence (ζ_{k_p}).

Let ζ_0 be the limit point of the convergent subsequence (ζ_{k_p}). Show that $|\zeta_0| = 1$.

Problem 3. We consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$\forall\, (x, y) \in \mathbf{R}^2 : f(x, y) = 2x^2 + xy^2$$

and the correspondence $F : \mathbf{R} \to \mathbf{R}$ defined by the rule

$$F(x) = \begin{cases} [-1, x], & \text{if } x \geq 0 \\ [-2, 0], & \text{if } x < 0 \end{cases}.$$

(1) Show that the correspondence F does not have the closed graph property.

(2) Show that the maximum value function $V : \mathbf{R} \to \mathbf{R}$ given by

$$\forall\, x \in \mathbf{R} : V(x) = \max\{f(x,y) : y \in F(x)\}$$

is well defined and find an algebraical rule of V.

(3) Show that the maximum value function V is continuous.

(4) Determine the maximum value correspondence $Y^* : \mathbf{R} \to \mathbf{R}$ given by

$$\forall\, x \in \mathbf{R} : Y^*(x) = \{y \in F(x) : V(x) = f(x,y)\}.$$

Problem 4. We consider the function $F : \mathbf{R}^3 \to \mathbf{R}$ given by the rule

$$\forall\, (t,x,y) \in \mathbf{R}^3 : F(t,x,y) = (x + y^2)e^{-t}.$$

Furthermore we consider the functional

$$I(x) = \int_0^1 \Big(x + \big(\frac{dx}{dt}\big)^2\Big)e^{-t}\, dt.$$

(1) Show that for every $t \in \mathbf{R}$ the function $F = F(t,x,y)$ is convex in $(x,y) \in \mathbf{R}^2$.

(2) Solve the variational problem: Determine the minimum function $x^* = x^*(t)$ of the functional $I(x)$ subject to the conditions $x^*(0) = 1$ and $x^*(1) = -\frac{1}{2}$.