Grading guide, Pricing Financial Assets, August 2014

1. Consider an economy over two time periods, $\mathcal{T} = \{0, 1, 2\}$, described by a recombining binomial tree. Two securities are traded: A non-dividend-paying stock with a price S_t that in each period can move up by a factor u (an up-move) or down with a factor d (a down-move), and risk free asset with a price B_t that will accumulate by a factor of e^r in each period. Assume $u > \mathsf{e}^r > d > 0$, $S_0 > 0$, and $B_0 = 1$.

   (a) Consider a European put option written on the stock with expiry at $t = 2$ and strike X. Use an arbitrage argument to find the value of the put at $t = 0$ expressed using a risk-neutral probability q of an up-move. Explain why the real-world probability p of an up-move does not enter directly in the expression.

   (b) Find the state prices (or Arrow-prices) at $t = 0$ for payments at $t = 1$

   (c) Now consider an American put in this model. Find and comment on a (non-trivial) example for the value of the American put to exceed the value of the European put.

Solution:

(a) Cf. Hull Chapter 12 and Hansen. A local arbitrage argument (which should be reproduced) gives that you can price any security f by discounting the properly calculated expected value of values (including potential payouts) one period ahead

$$f = \mathsf{e}^{-r}[qf_u + (1-q)f_d]$$

where

$$q = \frac{\mathsf{e}^r - d}{u - d}$$

Using backward induction and the fact that the payment of the put at $t = 2$ is $\max[0; X - S_2]$ you can find an explicit formula as (12.10) in Hull p.261. The real world probability does not enter directly into the formula because that information is already embedded in the prices of the traded instruments that do enter into the formula.

(b) The state prices are the discounted risk neutral probabilities i.e. $\psi_u = \mathsf{e}^{-r}q$ and $\psi_d = \mathsf{e}^{-r}(1-q)$ (thus $f = \psi_u f_u + \psi_d f_d$)

(c) We construct an example where the American option is optimally utilized before expiry, i.e. at $t = 1$. For a non-trivial case look at a situation where the option is in-the-money (ITM) in the down-state and out-of-the money (OTM) in the up-state at $t = 1$. The continuation value of the put is

$$\mathsf{e}^{-r}[qf_{ud} + (1-q)f_{dd}]$$

whereas the early exercise value is

$$\max[0, X - S_d] = X - S_d > 0$$

by our assumption. So early exercise is optimal if

$$X - S_d > \mathsf{e}^{-r}[qf_{ud} + (1-q)f_{dd}]$$
$$\mathsf{e}^r[X - dS_0] > [qf_{ud} + (1-q)f_{dd}]$$

If it is certain that option will be ITM at both the down-up and the down-down states at $t = 2$ then it is optimal to exercise now (at $t = 1$) because of discounting. To see this use that also

$$\mathsf{e}^r dS_0 = qudS_0 + (1-q)d^2 S_0$$

such that the above reduces as

$$\begin{aligned} \mathsf{e}^r[X - dS_0] &> q(X - udS_0) + (1-q)(X - d^2 S_0) \\ \mathsf{e}^r X &> X \end{aligned}$$

when $r > 0$.

We can construct an example where the option is ITM at the down-down but OTM at the down-up at $t = 2$ if

$$\begin{aligned} \mathsf{e}^r[X - dS_0] &> (1-q)(X - d^2 S_0) \\ (\mathsf{e}^r - 1)[X - dS_0] &> (1-q)(X - d^2 S_0) - (X - dS_0) \\ (\mathsf{e}^r - 1) &> (1-q)\frac{X - d^2 S_0}{X - dS_0} - 1 \end{aligned}$$

Since the left-hand side is positive we can choose d close to 1 (but below e^r) where the inequality will be satisfied (note that q and $1-q$ will be valid probabilities for this choice). Such a choice gives an example wich combines a sufficiently high interest rate with a sufficiently low volatility which makes the early exercise optimal.

2. (a) In the one-factor Gaussian model of defaults the default of an obligor i is indicated by

$$x_i = a_i F + \sqrt{1 - a_i^2} Z_i$$

where F and Z_i, $i = 1, \dots, n$, have independent standard normal distributions. Give an interpretation of the model and its parameters. What is the (copula) correlation between x_i and x_j for some $j \neq i$?

(b) In this model the indicator x_i is compared to the probability $Q_i(T)$ that the obligor has defaulted before or at time T. Suppose that the obligor defaults at time t. Which value of the indicator x_i does this correspond to?

(c) Describe in general terms how a Monte Carlo simulation can be used to model correlated defaults given the individual default probabilities and (copula) correlations.

Solution:

(a) Cf. Hull p.538-40 Note that F is a common factor determining the covariance structure, whereas Z_i are ideosyncratic factors, and a_i are sensitivities to the common factor. Note that $Var[x_i] = a_i^2 + (1-a_i^2) = 1$ and $Cov[x_i, x_j] = a_i a_j$ using that F and Z_i are standard normal and independent. In implementation the a_i are often approximated as the correlation between i's equity return and some broad market index (i.e. $\mathbb{P}$-type correlations). They could also be inferred from e.g. prices of tranched CDOs (i.e. $\mathbb{Q}$-type correlations)).

(b) The indicator can be interpreted as corresponding to a value of $N^{-1}(Q_i(t))$ (where N is the standard normal distribution function), i.e. we investigate defaults on the inverse normal evaluated on the relevant percentile of the given cumulated probability of default such that an observation of a larger x_i than this critical value means a default before or at t.

(c) The Monte Carlo methodology implies that you sample x_is from a multivariate normal distribution with a given correlation matrix. Then the x_i are converted to corresponding time to defaults for the individual obligors i. This in turn can be used to produce risk measures (under the $\mathbb{P}$-measure) or valuations (in principle under the $\mathbb{Q}$-measure - even if this distinction is often ignored in practise)

3. (a) In the Cox-Ingersoll-Ross (CIR) Model the (instantaneous) short term interest rate r is described by the process:

$$dr = a(b - r)dt + \sigma\sqrt{r}dz$$

where a,b and σ are constants ($\sigma^2 < 2ab$), and dz a Wiener process. What does this mean for the behaviour of the short term interest rate?

(b) In this model the solution for the price of a zero-coupon bond can be written

$$P(t,T) = A(t,T)\mathrm{e}^{-B(t,T)r(t)}$$

Derive the duration of the bond.

(c) In some models of the short term interest rate r (e.g. the Hull-White model) the drift rate is made a function of calendar time. What is the purpose of the extra flexibility compared to the CIR (or Vasicek) type of models?

Solution:

(a) Cf. Hull p. 685-87. It should be noted that the model implies a mean reversion of the spot rate to a level of b at a rate of a. It should also be note that the volatility term means that the short term rate cannot go below 0.

(b) Cf. Hull p.686-7. The answer depends on the definition of duration applied. The version in Hull (p.687) is

$$D = -\frac{\frac{\partial P}{\partial r}}{P} = B(t,T)$$

(c) The time-dependent drift term in the Hull-White model is introduced to be able to incorporate a given, initial term structure, making the the values derived from it "arbitrage-free" in relation to the existing securities priced on the current term structure (assuming these to be arbitrage free). This is in contrast to the CIR and Vasicek models of the "Equilibrium"-type that put restrictions on the possible initial term structure (as it is a function of the current spot rate only).