Grading guide, Pricing Financial Assets, June 2014

1. (a) Define a forward contract on a stock.

   (b) Consider a stock with current price S_0 that does not pay dividends. Assume a constant continuously compounded interest rate of r. Using an arbitrage argument find the forward price F_0 at time 0 on a forward contract on the stock that matures at time T.

   (c) Consider the value of a forward contract with forward price K at time $t < T$ and assume that the stock price follows the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dz$$

   Using Ito's lemma find the process followed by the value of the forward contract. What will the drift be under the risk neutral measure?

   (d) Assume now that the stock pays a constant continuous dividend rate of q. What will the forward price be?

Solution:

(a)

Definition 0.1 (Forward Contract)**.** *A **Forward Contract** is an agreement to buy or sell an asset at some future time T for an agreed price F. There are no initial payments. The price F paid at T is called the forward price or the delivery price*

(b) Cf. e.g. Hull p. 312-3. The argument may be: Buy the stock spot financed by a loan repaid at T. The only cashflow on this will be the repayment $S_0 \mathrm{e}^{rT}$ at T, leaving you with the stock. An alternative is to enter into a forward contract with delivery at T. To eliminate arbitrage it must be that the forward price at 0 for delivery at T is $F_0 = S_0 \mathrm{e}^{rT}$

(c) For a forward contract with forward price K the value is $V_t = S_t - K\mathrm{e}^{-r(T-t)}$. Using Ito's lemma you get that the differential form for the value process for the forward:

$$dV_t = [\mu S_t - rK\mathrm{e}^{-r(T-t)}]dt + \sigma S_t dz$$

For a currently priced forward (i.e. where K gives the forward contract a zero net present value $K = F_t = \mathrm{e}^{r(T-t)} S_t$) we see that the drift is 0 under the risk neutral measure (by setting $\mu = r$), while a non-negative market value will grow at a rate r. It will be considered ok just to derive this for the currently priced forward.

(d) In the arbitrage argument above you get too much if you buy the stock paying dividend, but you can repay and reduce the interest payments continuously leaving only $S_0 \mathrm{e}^{(r-q)T}$ to repay at T. Alternatively consider the price of a synthetic stock paying no dividends and compare that to S_0 and go on from there.

2. (a) Let the probability of a borrower not defaulting at or before time t be given by $V(t)$. What is the probability that the borrower will default between time t and $t + \Delta t$ conditional on not being in default at time t? Use this to define the continuously compounded default hazard rate.

   (b) How, and under which assumptions, may we estimate the hazard rate from the interest rate spread on bonds issued by the borrower? Under what probability measure would we say this estimate is derived? Compare this to a hazard rate that is derived from default frequencies and recovery ratios published by a rating agency.

(c) In one model by Merton the value of a claim on a company with limited liability is modelled using a variation of the Black-Scholes-Merton option model. What is the option features embedded in such a claim? What parameters that are not directly observable must be determined to price the claim in this model? Comment on the model.

Solution:

(a) The conditional default probability from t to $t + \Delta t$ is

$$\frac{V(t) - V(t + \Delta t)}{V(t)}$$

Given $V(t)$ the hazard rate is the rate of decay of survivors. Assuming differentiability you may define it as

$$\lambda(t) = -\frac{\frac{\partial V(t)}{\partial t}}{V(t)}$$

This can be motivated by considering the above discrete time step and letting the time step approach zero.

(b) Under a risk neutral probability measure the hazard rate may be derived from prices of traded assets, e.g. bonds from the company in question. Assuming that the recovery rate R is known you may estimate the average hazard rate as the hazard rate that makes the discounted expected payoff (i.e. taking defaults into account) to be the price of a risk free bond. Assuming the risky bond pays a spread of s over the risk free bond for a given maturity you can put

$$\bar{\lambda} = \frac{s}{1 - R}$$

You may similarly bootstrap a hazard rate structure from a term structure of credit spreads.

This analysis should, however, be seen as conducted under a risk neutral ("$\mathbb{Q}$") measure, so that (λ, R) will not be the same as frequencies and averages published by rating agencies, real world probabilities ("$\mathbb{P}$") (Hull section 23.4-5).

(c) In the Merton-model the value of risky debt issued by a limited liability entity is modelled as the value of the assets of the entity less the value of the equity held by the owners of the company. Due to the limited liability the owners can be seen to hold a call option on the assets.

Assuming there is only one form of debt, a zero coupon debt of face value D maturing at time T, the Black-Scholes-Merton Model can be applied. The underlying asset for that model would be the value of the entity assets, which is typically unknown, as is it's volatility. From Ito's lemma you can derive a relationship between the equity volatility and the asset volatility. Assuming that the value of the equity and the equity volatility is known form markets, you have two equations in two unknowns that can be solved numerically for the value of the debt (Hull, section 23.6).

A problem with Merton's model is that values are assumed to follow Ito-processes, i.e. with continuous sample paths. This makes the likelihood of a default a short time step ahead very small (no "jump-to-default"') and credit spreads to go to zero as the maturity of risky bonds goes to zero, which is not in line with empirical evidence (which may be due e.g. to asset values that fundamentally jump in case of default or to asymmetric information between owners and creditors on the value of assets, revealed at default).

3. (a) Give a definition of an interest rate swaption (also called a swap option)

   (b) Give an argument that a receiver swaption can be seen as a call option on a properly defined fixed rate bond. What can you say about the coupon and principal of such a bond?

   (c) Let PS denote the value of a swaption to pay a fixed rate s_K and receive LIBOR between times T_1 and T_2, let RS denote the value of a swaption to receive a fixed rate of s_K and pay LIBOR between times T_1 and T_2, and let RFS denote the value of a forward starting swap that receives a fixed rate of s_K and pays LIBOR between times T_1 and T_2. Assume that there are no arbitrage opportunities and i) show that $PS + RFS = RS$ and ii) deduce that $PS = RS$ when s_K equals the current forward swap rate.

Solution:

(a) Cf. Hull p.659f

Definition 0.2 (Swaption)**.** *A payer (receiver) swaption or swap option gives the holder the right to enter into a payer (receiver) Interest Rate Swap in the future*

(b) Cf. Hull p. 660 If the fixed rate coupon on the bond is equal to the swap rate the principal is the par value of the bond (barring credit risks, liquidity considerations etc.). The receiver swaption can be seen as an option to receive this bond by delivery of the principal (which we , somewhat idealized, assume will be equal to the continuing value of the variable leg of the swap). We can note that both swaption and bond option will increase in value with lower longer rates.

(c) Consider the payments from T_1 to T_2 in the following two cases that depend on the realized level of the swap rate s_{T_1} at T_1 with maturity at T_2:

Case $s_K > s_{T_1}$	Payment
PS	Pays s_K and receives $LIBOR$
RFS	Receives s_K and pays $LIBOR$
RS	0

Case $s_K \leq s_{T_1}$	Payment
PS	0
RFS	Receives s_K and pays $LIBOR$
RS	Receives s_K and pays $LIBOR$

From the table it is seen that the payments for the payer swaption and the forward starting swap equals the payments on the reciever swaption. Thus barring arbitrage the equation must hold. If s_K is the current forward swap rate the value RFS is by definition zero.