Suggested Solutions to: Resit Exam, Fall 2014 Contract Theory, February 19, 2015

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Question 1: Moral hazard with mean-variance preferences

(a) Solve for the β parameter in the second-best optimal contract, denoted by β^{SB} (you do not need to solve for α^{SB}, and you will not get any credit if you nevertheless do that). You should make use of the following (well-known) result:

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

- P chooses the parameters in the contract, α and β. In addition, P can effectively choose A's effort e, because P designs the incentives that A faces when deciding what effort to make. We can thus think of P as choosing α, β, and e in order to maximize his expected payoff, subject to A's individual rationality (IR) constraint and incentive compatibility (IC) constraint. P's problem:

$$\max_{\alpha,\beta,e}\left\{\overbrace{(1-\beta)\,e-\alpha}^{=EV}\right\}$$

subject to

$$\overbrace{-\int_{-\infty}^{\infty}\exp\left[-r\left(t-c\left(e\right)\right)\right]f\left(z\right)dz}^{=EU} \geq -\exp\left[-r\widehat{t}\right], \tag{IR}$$

$$e \in \arg\max_{e'} EU\left(e'\right). \tag{IC}$$

The IC constraint says that e indeed maximizes A's utility among all the e's that A could choose. The IR constraint says that A's expected utility if accepting the contract is at least as large as his utility from his outside option; this therefore ensures that A wants to participate.

- The IC constraint above is actually a whole set of infinitely many constraints. In order to reduce these to one single IC constraint, we can make use of the first-order approach, which means that we replace IC above with the first-order condition from A's maximization problem (for some arbitrary values of the contract parameters α and β). From the question we have that A's expected utility can be written as

$$EU = -\exp\left[-r\left(\alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2\right)\right].$$

Maximizing EU is equivalent to maximizing a monotone transformation of this expression, so we can without loss of generality let A maximize

$$\widetilde{EU} = \alpha + \beta e - \frac{1}{2}e^2 - \frac{1}{2}\nu r \beta^2. \tag{1}$$

- We have

$$\frac{\partial \widetilde{EU}}{\partial e} = \beta - e = 0$$

Therefore A's optimal effort level is

$$e = \beta. \tag{2}$$

- We can write the IR constraint as

$$
-\int_{-\infty}^{\infty} \exp\left[-r\left(t-c\left(e\right)\right)\right] f\left(z\right) dz \geq
$$

$$
-\exp\left[-r\widehat{t}\right] \Leftrightarrow
$$

$$
-\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] \geq
$$

$$
-\exp\left[-r\widehat{t}\right] \Leftrightarrow
$$

$$
\exp\left[-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right)\right] \leq
$$

$$
\exp\left[-r\widehat{t}\right] \Leftrightarrow
$$

$$
-r\left(\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2\right) \leq -r\widehat{t} \Leftrightarrow
$$

$$
\alpha+\beta e-\frac{1}{2}e^2-\frac{1}{2}\nu r\beta^2 \geq \widehat{t} \Leftrightarrow
$$

$$
\alpha \geq \widehat{t}-\beta e+\frac{1}{2}e^2+\frac{1}{2}\nu r\beta^2
$$

Plugging in (2) in this inequality, we obtain

$$
\begin{aligned}
\alpha &\geq \widehat{t}-\beta^2+\frac{1}{2}\beta^2+\frac{1}{2}\nu r\beta^2 \\
&= \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2.
\end{aligned}
$$

Plugging in (2) into P's objective function $EV=\left(1-\beta\right)e-\alpha$, we have

$$
EV=\left(1-\beta\right)\beta-\alpha.
$$

- Using the above results, P's problem becomes

$$
\max_{\alpha,\beta}\left\{\left(1-\beta\right)\beta-\alpha\right\} \qquad \text{subject to}
$$

$$
\alpha \geq \widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2. \qquad \text{(IR)}
$$

- It is clear that IR must bind, as the objective is decreasing in α and the constraint is tightened as α is lowered (thus P wants to lower α until the constraint says stop). We thus have $\alpha=\widehat{t}-\frac{1}{2}\left(1-\nu r\right)\beta^2$. Plugging this value of α into the objective yields the following unconstrained problem:

$$
\boxed{\max_\beta\left\{\beta-\tfrac{1}{2}\left(1+\nu r\right)\beta^2-\widehat{t}\right\},}
$$

with the first-order condition

$$
1-\left(1+\nu r\right)\beta=0 \Rightarrow \beta^{SB}=\frac{1}{1+\nu r}.
$$

(b) Does the agent get any rents at the second-best optimum? Do not only answer yes or no, but also explain how you can tell.

- No, he does not get any rents at the second-best optimum. "Rents" are defined as any payoff from accepting the contract that exceeds the outside option payoff. However, we saw under a) that the IR constraint binds at the optimum, which means that A does not get any rents.

(c) The first-best values of the effort level and the β parameter equal $e^{FB}=1$ and $\beta^{FB}=0$, respectively. How do these values relate to the corresponding second-best values? In particular, is there under- or overprovision of effort at the second-best optimum?

- We have from the above analysis that $\beta^{SB}=e^{SB}=\frac{1}{1+\nu r}$. We see that there is underprovision of effort (as $e^{SB}<e^{FB}$). We also see that the beta-parameter is too large relative to the first best level ($\beta^{SB}>\beta^{FB}$).

(d) Consider the limit case where $r\rightarrow 0$. Explain what happens to the relationship between the second-best and the first-best effort levels. Also explain the intuition for this result.

- In the limit where $r\rightarrow 0$, A is risk neutral. We see from above that in that limit, $e^{SB}=1$. That is, the second-best effort level coincides with the first-best level: there is no inefficiency in spite of the fact that there is asymmetric information. The reason why this can occur is that when A is risk neutral he doesn't mind bearing risk. Therefore P can incentivize A very strongly, so that indeed $\beta^{SB}\rightarrow 1$ as $r\rightarrow 0$: A's compensation depends fully on the stochastic variable, so he makes the same decision as P would have made if he had been in A's job.
- The intuition is the same as we have discussed in other parts of the course, for example in the 2x2 moral hazard model with a risk neutral agent who is not protected by limited liability. There we explained the intuition as follows:
  - The economic meaning of the fact that A is risk neutral is that he cares only about

whether his payment t is large enough *on average*. Hence, P can, without violating the participation constraint, incentivize A by giving him a negative payment (in practice a penalty) in case of a low output. More generally, P can achieve the first-best outcome by making A the residual claimant:

* Then A effectively buys the right to receive any returns: "the firm is sold to the agent".
* Thereby, the effort level is chosen by the same individual who bears the consequences of the choice.
* In this situation A makes the same effort choice as P would have made.

Question 2: Consumer learning in an insurance market

(a) Explain briefly in words what each one of the seven constraints says and why the constraints are required if P wants to induce information gathering, interact with both types and offer them distinct contracts.

- IR-high, IR-low, IC-high, IC-low are the usual constraints from the standard setup with an exogenous information structure. They ensure that A does what P wants at the interim stage (i.e., at the point in time when each agent type has learned its own type). That is, IR-high (IR-low, resp.) ensures that the high-type (low-type, resp.) agent prefers the contract aimed at him to his outside option. And IC-high (IC-low, resp.) ensures that the high-type (low-type, resp.) agent prefers the contract aimed at him to the contract aimed at the other type.

- IR-ante ensures that A, at the point in time when he has not yet learned his type, prefers to incur the information gathering cost and then obtain his expected utility from the insurance contract (given that he chooses the contract that P wants him to choose), rather than not incurring the cost, leaving the interaction with P and instead get his outside option payoff.

- IG-low (IG-high, resp.) ensures that A, at the point in time when he has not yet learned his type, prefers to incur the information gathering cost and then obtain his expected utility from the insurance contract (given that he chooses the contract that P wants him to choose), rather than not incurring the cost, continue to interact with P and choose the low type's (high type's) contract.

(b) Show that IG-low and IR-low bind at the optimum.

First consider the first-order condition with respect to $\overline{u}_N$:

$$\frac{\partial \mathcal{L}}{\partial \overline{u}_N} = 0 \Leftrightarrow (1-\upsilon)\left(1-\overline{\theta}\right) h'\left(\overline{u}_N\right) = \underline{\mu}\,(1-\upsilon)\left(1-\overline{\theta}\right) - \overline{\mu}\upsilon\,(1-\underline{\theta})\,. \quad (3)$$

Given $h' > 0$ and $\overline{\mu} \geq 0$, the first claim that $\underline{\mu} > 0$ (which means that IG-low binds) follows immediately from (3). Now consider the second claim that $\lambda > 0$ (which means that IR-low binds). The first-order condition with respect to $\underline{u}_N$ is

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_N} = 0 \Leftrightarrow \upsilon\,(1-\underline{\theta})\, h'\left(\underline{u}_N\right) = \lambda\,(1-\underline{\theta}) + \overline{\mu}\upsilon\,(1-\underline{\theta}) - \underline{\mu}\,(1-\upsilon)\left(1-\overline{\theta}\right). \quad (4)$$

Now, add (3) and (4):

$$(1-\upsilon)\left(1-\overline{\theta}\right) h'\left(\overline{u}_N\right) + \upsilon\,(1-\underline{\theta})\, h'\left(\underline{u}_N\right) = \lambda\,(1-\underline{\theta})\,.$$

This equality (using $h' > 0$) implies that $\lambda > 0$. □

(c) Show that the $\underline{\theta}$ type is underinsured ($\underline{u}_N > \underline{u}_A$) at the optimum.

The first-order condition with respect to $\underline{u}_A$ is

$$\frac{\partial \mathcal{L}}{\partial \underline{u}_A} = 0 \Leftrightarrow \upsilon\underline{\theta} h'\left(\underline{u}_A\right) = \lambda\underline{\theta} + \overline{\mu}\upsilon\underline{\theta} - \underline{\mu}\,(1-\upsilon)\,\overline{\theta}. \quad (5)$$

To prove the claim that $\underline{u}_N > \underline{u}_A$, multiply (4) by $\underline{\theta}$ and multiply (5) by $(1-\underline{\theta})$:

$$\upsilon\underline{\theta}\,(1-\underline{\theta})\, h'\left(\underline{u}_N\right) = \lambda\underline{\theta}\,(1-\underline{\theta}) + \overline{\mu}\upsilon\underline{\theta}\,(1-\underline{\theta}) - \underline{\mu}\,(1-\upsilon)\,\underline{\theta}\left(1-\overline{\theta}\right),$$

$$\upsilon\underline{\theta}\,(1-\underline{\theta})\, h'\left(\underline{u}_A\right) = \lambda\underline{\theta}\,(1-\underline{\theta}) + \overline{\mu}\upsilon\underline{\theta}\,(1-\underline{\theta}) - \underline{\mu}\,(1-\upsilon)\,\overline{\theta}\,(1-\underline{\theta})\,.$$

Next subtract the second one of those equalities from the first one and simplify:

$$\upsilon\underline{\theta}\,(1-\underline{\theta})\left[h'\left(\underline{u}_N\right) - h'\left(\underline{u}_A\right)\right] = \underline{\mu}\,(1-\upsilon)\left(\overline{\theta}-\underline{\theta}\right). \quad (6)$$

The claim now follows from the strict convexity of h and the result in part (b) that $\underline{\mu} > 0$. □

(d) Suppose that it is indeed optimal for P to induce information gathering. Then what is the effect on P's profits, at the optimum, of an exogenous increase in the information gathering cost c? Will such an increase make P's (optimized) profits increase or decrease, or are the profits unaffected by a change in c? Do not show any calculations, but explain in words the reasoning behind your answer.

By the envelope theorem we know that the optimized value of P's profits will be affected by a change in the cost c only through the direct effect through c — all the indirect effects via the endogenous variables are zero at the optimum. Moreover, in P's optimization problem, the parameter c does not appear in the profit expression and it appears only in three of the constraints: in IR-ante, IG-low and IG-high. In particular, each one of these three constraints becomes more stringent as c becomes larger (the reason is that P wants to induce A to gather information, which obviously is more costly to do if c is large). Therefore, if anyone of these constraints is binding at the optimum, then the increase in c has a strictly negative impact on P's maximized profit. Indeed, we know from the (b) question that IG-low binds at the optimum. We can thus conclude that an increase in c makes P strictly worse off.

That is, in words, the reason why P is worse off from an increase in c is that such an increase makes A less willing to gather information, which means that it becomes more expensive for P to induce A to indeed gather information.