Written Exam at the Department of Economics

Summer 2018

Derivatives Pricing

SOLUTION GUIDE

Final Exam

May 31, 2018

3 hours, open book exam

Answers only in English.

This exam question consists of 5 pages in total

NB: If you fall ill during an examination at Peter Bangsvej, you must contact an invigilator in order to be registered as having fallen ill. In this connection, you must complete a form. Then you submit a blank exam paper and leave the examination. When you arrive home, you must contact your GP and submit a medical report to the Faculty of Social Sciences no later than seven (7) days from the date of the exam.

Be careful not to cheat at exams!

You cheat at an exam, if during the exam, you:

- Make use of exam aids that are not allowed
- Communicate with or otherwise receive help from other people
- Copy other people's texts without making use of quotation marks and source referencing, so that it may appear to be your own text
- Use the ideas or thoughts of others without making use of source referencing, so it may appear to be your own idea or your thoughts
- Or if you otherwise violate the rules that apply to the exam

Guidelines:

- The exam is composed of 4 problems, each carrying an indicative weight.
- If you lack information to answer a question, please make the necessary assumptions.
- Please clearly state any assumptions you make.
- All answers must be justified.

The students are expected to deliver relatively short answers to the questions, addressing key points presented in the course.

In questions that require calculation the method/approach used to obtain the result must be clear from the answer.

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You have just been hired as an analyst on the options trading desk in a large Nordic investment bank. Your new boss has a number of problems he needs your help with. Eager to show the senior traders that you know your stuff (as you have followed the course in Derivatives Pricing), you immediately start working on the problems.

Problem 1 (25%)

One of the traders has just sold a 1-year vanilla call option with strike 90. Assume the Black-Scholes assumptions are satisfied, the current stock price $S_0 = 90$, interest rates and dividends are zero, and the stock volatility $\sigma = 20\%$. The trader sold the option at its BSM price and continuously delta-hedges it using an implied volatility of 20%.

a) If the instantaneous realized volatility is a constant 20% and the terminal stock price is $S_T = 90$, what is the final PnL of the delta-hedged call option in one year?
We are in the BSM setting and our continuous delta-hedge position perfectly replicates the option. The net PnL is therefore deterministic and equal to zero.

Now, let us see what happens to the final PnL when we start relaxing the BSM assumptions. We first investigate what happens if realized volatility is no longer equal to the volatility used to calculate the delta hedge:

b) If the instantaneous realized volatility is a constant 10% and the terminal stock price is $S_T = 90$, what can you say about the final PnL of the delta-hedged call in one year?
The trader sold the option at a higher volatility than what is being realized. Thus, the expected PnL is positive. Nevertheless, the PnL is path-dependent. The largest final PnL is obtained along the paths that stay close to the strike as the option's gamma is the highest in this region.

c) If the instantaneous realized volatility is initially low and then high with the final root-mean square volatility of 20% and the terminal stock price is $S_T = 90$, what can you say about the final PnL of the delta-hedged call in one year?
Again, the final PnL is path-dependent. The largest final PnL is obtained along the paths that stay close to the strike in the beginning when volatility is below 20% and far away from the strike when volatility is above 20%. As Gamma gets higher when the option approaches expiry, we would expect the final PnL to be negative along paths close to the strike.

d) Would the final PnL in questions **a), b) and c)** be any different if the final stock price turns out to be $S_T = 120$?
(a) The answer is unchanged as we perfectly replicate the option and the final PnL is deterministic. In both (b) and (c) the PnL is path-dependent and thus depends on the price path of the underlying stock. The further away the stock price is from the strike, the smaller PnL contribution we get from realized volatility.

e) If the client does not delta-hedge the call option, explain under what scenarios (i.e., combinations of final stock price and realized volatility) you would both end up making money on the trade.
The trader sold the option, so he would make money if either realized volatility is less than the implied volatility of 20% she sold the option at. She gets the largest PnL along the path when the stock price stays close to the strike as gamma is highest in that region. The client doesn't care about the realized volatility as he doesn't delta hedge. He will profit only if the option ends up in-the-money, i.e. if the stock price is higher than the strike of 90. Therefore, you may both profit, if the option ends in-the-money and the realized volatility has been below implied volatility.

In practice, continuous delta hedging is usually not feasible. Instead, traders hedge their positions at discrete time intervals.

f) Why is continuous delta hedging not feasible in practice?
The primary obstacles are transaction costs and lack of liquidity. Physical constraints are also an issue, but with the development of high-speed broadband and fast computers, we can get close.

g) How would discrete delta hedging affect your conclusion in question **a)**?
The expected PnL is still zero but PnL is no longer deterministic but forms a distribution around zero.

Problem 2 (25%)

Assume the price of a non-dividend paying tradable asset X_t follows the risk-neutral dynamics

$$dX_t = \tilde{\sigma} dW_t$$

where $\tilde{\sigma}$ is a positive constant.

a) Why is this not a reasonable model for the price of a stock?
Stock owners of a listed company have limited liability. This means the stock price is bounded at zero. In this model, X is not bounded at zero and can go negative. Therefore, it is not an appropriate model for a stock price. (The model is, however, commonly used to model interest rates).

b) How is the interpretation of the volatility $\tilde{\sigma}$ different in this model compared to the volatility in the BSM model?

In the BSM model, volatility is expressed relative to the asset price. This means, volatility is fixed in relative terms but variable in absolute terms – the higher the asset price, the larger the absolute volatility. In the present model, however, volatility is expressed in absolute terms. This means, the level of absolute volatility is independent of the asset price.

c) Show that you can rewrite this model as a local volatility model with risk-neutral dynamics given by

$$\frac{dX_t}{X_t} = \sigma(X_t)dW_t$$

By defining the local volatility function as

$$\sigma(X_t) := \frac{\tilde{\sigma}}{X_t}$$

we can rewrite the model as a local volatility model.

d) Find an approximation of implied volatility as a function of the strike by using that implied volatility is roughly the average of the local volatilities between the current spot price and the strike.

The simplest (yet rough) approximation, is just to take the average of the local vol at the spot price X_0 and at the strike K

$$\Sigma(K) = \frac{1}{2}[\sigma(X_0) + \sigma(K)] = \frac{\tilde{\sigma}}{2}[X_0^{-1} + K^{-1}]$$

Some students may derive more precise approximations, and these are, of course, just as valid answers.

e) Show that the implied volatility skew is negative in this model.

Simply differentiate the implied vol approximation of the previous question wrt. strike

$$\frac{\partial \Sigma(K)}{\partial K} = -\frac{\tilde{\sigma}}{2K^2}$$

As the sign is negative, we have a downward sloping implied volatility smile

f) Discuss why equity index options usually exhibit a negative implied volatility skew.

Investors, who are usually long the equity index, are predominantly worried about hedging against large losses. Buying OTM puts gives them the necessary protection and they are willing to pay a premium for that. This causes OTM puts to be more expensive than OTM calls, and the implied vol skew to be negative. From a more fundamental viewpoint, the true underlying stock dynamics is not a Geometric Brownian Motion. Equity index returns are not normally distributed; their volatility is non-constant and tends to spike up when the market drops (the leverage effect).

To use the model for option pricing, you need to calibrate it to market prices. Therefore, you have obtained the following mid prices of call options on the asset X_t from your broker:

Strike	Expiry (years)	Call price
95	1.00	10.727
100	1.00	7.979
105	1.00	5.727
100	1.05	8.176

The current spot price is $X_0 = 100$ and let the riskless interest rate be zero.

g) By valuing a calendar and butterfly spread, approximate the at-the-money local volatility in one year using Dupire's equation and estimate the volatility $\tilde{\sigma}$.

The prices of the calendar and butterfly spreads are:

$$Calendar = 8.176 - 7.979 = 0.1979$$
$$Butterfly = 10.727 - 2 * 7.979 + 5.727 = 0.4961$$

And from these we can approximate the derivatives

$$\frac{\partial C}{\partial T} \approx \frac{\text{Calendar}}{\text{dT}} = \frac{0.1979}{0.05} = 3.9408$$
$$\frac{\partial^2 C}{\partial K^2} \approx \frac{Butterfly}{dK} = \frac{0.4961}{5^2} = 0.0198$$

Next, we can approximate the local vol in K=100, T=1.00 using Dupire's equation

$$\sigma^2(K,T) = \frac{2\frac{\partial C}{\partial T}}{K^2\frac{\partial^2 C}{\partial K^2}} \approx \frac{2 * 3.9408}{100^2 * 0.0198} = 0.040 \quad \Rightarrow \quad \sigma(100,1) \approx 0.20$$

From the local vol function we have

$$\tilde{\sigma} = K\,\sigma(K) \approx 100 * 0.20 = 20$$

Your colleague has just sold a 1-year at-the-money call option on X_t at an implied volatility of 20%. The current spot price is 100 and $\tilde{\sigma} = 20$.

h) If she delta-hedges the option using the BSM model based on the implied volatility, has she then over- or under-hedged the call option relative to the delta in the local volatility model?

She has over-hedged, i.e. taking a too large position in the underlying asset. For a small increase in the underlying asset, the model predicts that the call price will increase less than in the BSM model as the volatility decreases due to the negative skew. Formally, the local volatility delta is approximately less than the BSM delta if the implied vol skew is negative

$$\Delta^{\text{LV}} \approx \Delta^{\text{BSM}} + \nu^{\text{BSM}}\frac{\partial \Sigma}{\partial \text{K}}$$

where ν^{BSM} is the BSM vega which is positive for puts and calls

Problem 3 (20%)

Assuming zero interest rates and dividends, the index price S_t is governed by the Heston model with risk-neutral dynamics

$$\frac{dS_t}{S_t} = \sqrt{V_t}dW_t$$
$$dV_t = -\lambda(V_t - \bar{v})dt + \eta\sqrt{V_t}dZ_t$$
$$dW_t dZ_t = \rho dt$$

where $V_0, \lambda, \bar{v}, \eta, \rho$ are parameters of the model.

a) What is the impact of stochastic volatility on the implied volatility surface?

If we let the instantaneous volatility (or variance) be stochastic, OTM and ITM option prices get extra value from the convexity in volatility (the Volga effect). The risk-neutral distribution of the stock price has fatter tails. As a result, the model will generate a symmetric implied volatility smile.

b) Explain how the speed of mean-reversion in volatility λ and the correlation ρ impact the implied volatility surface.

The speed of mean reversion determines how fast the instantaneous variance converges to its long-run mean $\bar{v}$. Mean-reversion makes the implied volatility smile flatten out for longer expirations. The higher the speed of mean reversion, the earlier (i.e., shorter expirations) the implied volatility smile starts to flatten. The correlation parameter determines the correlation between the instantaneous variance and the stock price. A negative correlation will generate a negative implied volatility skew, while with a correlation of zero, one gets a symmetric implied volatility smile.

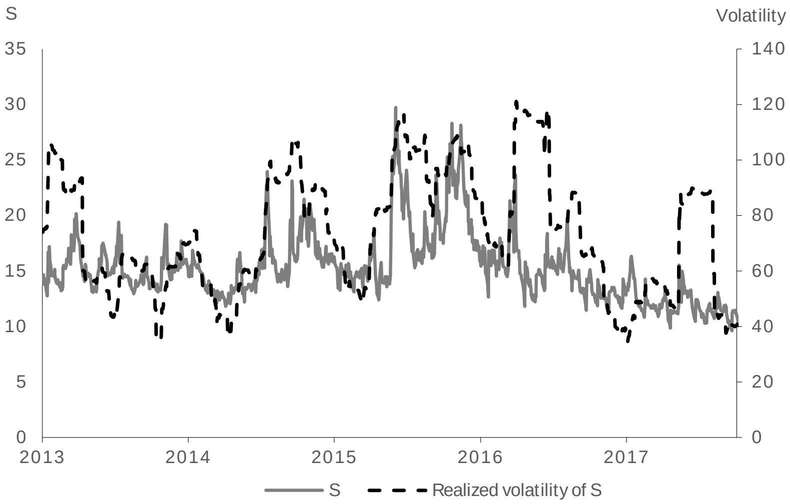

Figure 1: the historical prices of S_t and its 60-days' realized volatility

c) Based on the historical data on S_t in Figure 1, which sign would you expect ρ to have?

Volatility tend to spike up when the S increases. Therefore, one would expect ρ to be positive. In equity markets, we usually observe the opposite pattern, volatility increases when the underlying drops.

d) Why do stochastic volatility models have difficulties generating the steep short-term skew observed in equity options markets?

In a pure stochastic volatility model, the instantaneous volatility (or variance) is driven by a continuous diffusion process. This means that for short expirations the volatility cannot have diffused too far from its initial value. It would require a very high vol-of-vol to account for the steep short-term skew of equity options.

e) Devise a trading strategy in terms of vanilla options that will profit if volatility of volatility (vol-of-vol) increases but is neutral to the overall level of volatility.

Several strategies can be proposed here. A simple choice is to buy a strangle and sell an ATM straddle. The size of the two trades should be set so the total strategy is vega neutral. This means you should sell relatively less of the ATM straddle as it has larger vega exposure. Dynamically hedging this strategy will leave you exposed to vol-of-vol.

Problem 4 (30%)

Assume the Black-Scholes assumptions are satisfied. For simplicity, let the riskless interest rate be zero and assume the underlying stock pays zero dividends.

The BSM model assumes the risk-neutral dynamics of underlying asset follows a geometric Brownian motion:

$$\frac{dS_t}{S_t} = \sigma dW_t$$

a) Derive the risk-neutral dynamics of $\log(S_t)$

By Ito's lemma, one can obtain the dynamics:

$$d\log(S_t) = -\frac{1}{2}\sigma^2 dt + \sigma dW_t$$

b) If the current stock price is 80 and the stock's volatility is 20% per year, what is the risk-neutral probability that it ends above 150 in 1 year?

$$\begin{aligned} P(S_T > 150) &= 1 - P(S_T \leq 150) = 1 - P(ln(S_T) \leq ln(150)) \\ &= 1 - N\left(\frac{ln(150) - \left(\ln(S_0) - \frac{1}{2}\sigma^2 T\right)}{\sigma\sqrt{T}}\right) \\ &= 1 - \mathrm{N}\left(\frac{ln(150) - \left(ln(80) - \frac{1}{2} * 0.20^2 * 1\right)}{0.20 * \sqrt{1}}\right) \approx 0.059\% \end{aligned}$$

One of the senior trader asks you to price a European-style exotic derivative contract to an important client. The payoff at expiration *T* of this contract is

$$\max(S_T^2 - K^2, 0)$$

where *K* is the strike and S_T is the terminal stock price

c) Derive an analytical formula for the contract's price in the BSM model.

There're different ways to derive this. Probably the simplest approach is to observe the fact that:

$$S_T = S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma W_T} \quad \Rightarrow \quad S_T^2 = (S_0^2 e^{\sigma^2 T}) e^{-\frac{1}{2}(2\sigma)^2 T + (2\sigma) W_T}$$

This means, we can price the exotic option using the BSM formula with a spot price of $S_0^2 e^{\sigma^2 T}$, a strike of K^2 and a volatility of 2σ. Inserting into the BSM formula for a vanilla call, we obtain an analytical formula for the exotic price

$$V(S_0, K) = S_0^2 e^{\sigma^2 T} \mathrm{N}\left(\frac{\ln(S_0/K)}{\sigma\sqrt{T}} + \frac{3}{2}\sigma\sqrt{T}\right) - K^2 N\left(\frac{\ln(S_0/K)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right)$$

d) If the current stock price is 80, *K*=100, the stock's volatility is 20% per year, and time to expiration is 1 year, what is the price of this contract in the BSM model?

By inserting in the formula from the previous question, one obtains

$$V(S_0,K) = 80^2 * e^{0.20^2*1} * \mathrm{N}\left(\frac{\ln(80/100)}{0.20*\sqrt{1}} + \frac{3}{2}*0.20*\sqrt{1}\right) - 100^2 * N\left(\frac{\ln(80/100)}{0.20*\sqrt{1}} - \frac{1}{2}*0.20*\sqrt{1}\right)$$
$$\approx 260.61$$

Your senior colleague disagrees with your pricing of the exotic contract. She does not believe the BSM assumptions and argues that you need to use a more advanced model that fits the implied volatility smile.

e) Show that the exotic contract can be statically replicated with a portfolio of call options.
As the exotic contract has a European payoff, we can statically replicate the contract using vanilla options.
Let $V(S_T)$ denote the payoff function in terms of the Heaviside function H

$$V(S_T) = max(S_T^2 - K^{2,}0) = (S_T^2 - K^2)\mathrm{H}(S_T^2 - K^2)$$

The partial derivatives are

$$\frac{\partial V(S_T)}{\partial S_T} = 2S_T\ \mathrm{H}(S_T^2 - K^2) + 2S_T\ (S_T^2 - K^2)\delta(S_T^2 - K^2) = 2S_T\ \mathrm{H}(S_T^2 - K^2)$$

$$\frac{\partial^2 V(S_T)}{\partial S_T^2} = 2\mathrm{H}(S_T^2 - K^2) + 4S_T^2\delta(S_T^2 - K^2)$$

The exotic contract price can then by replicated in terms of call options C(S,K) as

$$V(S_0,K) = \int_K^\infty \frac{\partial^2 V(x)}{\partial x^2} C(S_0,x)dx = \int_K^\infty [2H(x^2 - K^2) + 4x^2\delta(x^2 - K^2)]C(S_0,x)dx$$
$$= 4\int_K^\infty x^2\delta(x^2 - K^2)C(S_0,x)dx\ +\ 2\int_K^\infty H(x^2 - K^2)C(S_0,x)dx$$
$$= 4K^2C(S_0,K) + 2\int_K^\infty C(S_0,x)dx$$

Thus, the exotic contract depends on the implied volatility smile from strike K and out.

Assume you have calibrated two different stochastic volatility models; a model with mean-reversion in volatility and a model without mean-reversion in volatility. It turns out, however, that both models exactly fit the market prices of vanilla options with expiration at time *T* and therefore completely match the observed implied volatility smile.

f) Do the two models give the same or different prices of the exotic contract?
As the exotic contract has a European payoff it's price only depends on the terminal risk-neutral distribution of S_T. Therefore, it can be statically replicated in terms of vanilla options. As both models perfectly match the prices of vanilla options, they will also produce the same price of the exotic contract.

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