Herd Behaviour in Efficient Financial Markets with Sequential Trades

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Abstract

We describe conditions on signal distributions that are necessary and sufficient for informational herding in a stylized model of sequential specialist security trading. Curiously, there can be persistent herding even with signals that satisfy the Monotone Likelihood Ratio Property. Price paths are strongly biased in the direction of the herd but prices are also very sensitive to movements against the herd. Price movements thus become more pronounced through herding. Numerical simulations indicate that the probability of herding and the level of noise trading are inversely related. Our results contrast the existing literature which found that herding with monotonic signals is impossible, and that herding is rarely accompanied by price movements. The paper thus allows a new perspective on herding in financial markets with efficient prices. We identify that the major ingredient needed for herding is that some agents find their information confusing.

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1 Introduction

When markets crash in the absence of any significant shifts in economic fundamentals, an explanation frequently heard is that investors behaved like a herd that stampeded without cause. But mass-uniform behavior need not be triggered by 'animal spirits' — it can be fully rational. Conventionally so called 'rational herding' occurs in situations with information externalities, when agents' private information is swamped by the information derived from observing others' actions. Such 'herders' deliberately act against their private information (but technically they do not discard it) and follow the crowd.

At first sight, this concept provides a tailor-made explanation for financial market frenzies, crashes and panics. In models of financial markets, however, prices are typically assumed to be informationally efficient, so that the current price reflects all public information. Suppose a crowd of people sells frantically. An investor with favourable information will update his information, and, indeed many sales will lower his expectation. At the same time, however, prices adjust downward too, so it is not clear that such an investor will now sell – to him the security may still be cheap. So for herding private expectations and prices must diverge substantially: Once favourable expectations must drop faster or unfavourable expectations must rise faster than prices.

In an important paper, Avery and Zemsky (1998), henceforth called AZ, argue that herd behavior with informationally efficient asset prices is not possible unless signals are "non-monotonic" and risk is "multi-dimensional".[1] In particular, they employ a specialist sequential security trading model à la Glosten and Milgrom (1985), and show the following:

• With two possible security-values and two signals there can be no herding.

• With three security-values, three signals and a special non-monotone information structure (with two-dimensional risk) herding is possible; however, there is very little price movement during the herd phase.

• If in addition to the above information structure, with three security-values and three signals, traders have different abilities to interpret the signals and this is private information then herding with extreme price movements (bubbles) is possible. (In their model the likelihood of large price movements during a herd phase is extremely small; of the order of $10^{-6}\times$probability of a particular sequence of trades.)

The profession, for instance Brunnermeier (2001), Bikhchandani and Sunil (2000), Chamley (2004) have derived three messages from this paper. First, with 'monotonic' signals, herding is impossible. Second, for herding one needs 'multidimensionality' of risk.

[1] Lee (1998) also uses a financial market model with moving prices. His herding, or rather 'information avalanches' result is, however, based on frictions induced by transaction costs. For a recent, comprehensive survey of the herding literature see Hirshleifer and Teoh (2003).

Third, herding does not involve violent price movements except in the most unlikely environments. Therefore, since in AZ the information structure is very special and in general large price movements cannot be explained by herd-type behavior, one can conclude that rational herding models are not so relevant to understanding the functioning of efficient financial markets.

We do not contest Avery and Zemsky's very insightful results, nor do we argue that our information structures have more intuitive appeal. But in this paper we argue the following two points.

• The profession's perception needs being corrected as one needs neither non-monotonic signals nor multidimensionality of risk. Even with "MLRP"-signals there may be a great deal more rational informational herding than is currently expected in the literature.

• Extreme price movements with herding are possible under not so unlikely situations.

In particular, we provide a necessary and sufficient condition for informational herding in the same stylized specialist sequential security trading model with three possible security-values, three signals,[2] and a symmetric prior distribution (we employ symmetry to reduce the number of potential sources for herding). We also assume that signals obey the Monotone Likelihood Ratio Property (MLRP).[3] Our signals do not resemble any form of 'multiple dimensions of uncertainty', the model is straightforwardly one-dimensional. We then argue that herding can occur if and only if for one signal the conditional signal distribution is U-shaped in liquidation values and the proportion of informed agents is not too large. Finding herding in this setup is, therefore, a strong result because our signals still satisfy the MLRP, which is in itself a strong concept. This contrasts with Avery and Zemsky, who employ a definition of signal-monotonicity that is non-standard. Indeed, if signals satisfy their monotonicity-condition there can be no herding (their monotonicity condition almost rules out herding by definition), but it is not clear though how their monotonicity concept relates to those commonly used in the literature. Furthermore, their monotonicity refers to the dynamic development of private and public expectations, not to primitives of the signal distribution.

In contrast to AZ's examples with little price movements during herding (except very exceptionally), we show in our setting (with MLRP and a single dimension of uncertainty) that prices can move substantially during herding: investors will continue to herd as long

[2]Our results in this paper are established with a discrete number of (three) signals, but the results are robust if signals are continuous. Details are available from the authors.

[3]The *Monotone Likelihood Ratio Property* is the standard signal monotonicity requirement in the literature, found for instance in rational expectations models or auctions. It is a convenient tool, as, for instance, investors' expectations are ordered so that higher signals imply higher expected liquidation values.

as trades are 'in the direction of the crowd'. Graphically speaking, as long as no-one stops, the lemmings keep walking off the cliff. Moreover, the range of herding-prices can comprise almost the entire range of feasible prices and during buy-herding (sell-herding), prices increase (decrease) stronger and decrease (increase) faster than if there was no herding possibility (where agents are "naïve" and rely only on their own private information).[4] Casual intuition suggests that in an informational-learning model the possibility of buy-herding (analogous arguments hold for sell-herding) should *(i)* hamper learning from buys so that the upward-trajectory of prices should become relatively flatter and *(ii)* carry a strong negative message when sales (which signify bad states of the world) occurs. The second assertion is true, but the first is incorrect: Despite herding 'buys' carry strongly favorable information. Therefore, the possibility of buy-herding makes price movements more volatile in the short-run compared with the case in which there is no herding.

There are some additional implications of herd behavior that we highlight in this paper. Since some types of traders change their trading modes when herding, prices are strongly history-dependent: More specifically, as the entry order of traders is pesrmutated, prices with the same population of traders can be strikingly different. Also, herding results in price paths that are very sensitive to changes in some key parameters. In particular, as we noted before, a necessary condition for herding is that the proportion of informed agents is below some critical level (enough noise traders). Comparing two situations, one with the proportion of informed agents just below the critical level to trigger herding and one with just above to prevent herding, prices deviate substantially in the two cases for an identical group of traders.

The focus of this paper is on herding and its effects, but we can also describe conditions for so-called "contrarian" behaviour: a trader acts as a contrarian if after observing a trading history he changes his action and acts against the general movement. A necessary and sufficient condition for such behaviour is that an agent has a "hill-shaped" conditional signal distribution. This observation then closes the circle of understanding the different kinds of behaviors triggered by conditional signal distributions. A U-shaped conditional signal distribution in essence confuses the recipient, because in his posterior he will place more weight on extreme values; this makes the signal recipient to go with the flow and thus makes herding possible. A hill-shaped signal distribution achieves the opposite — in his posterior the recipient will shift weight to the center. This makes the recipient of a hill-shaped signal more stubborn so that he may act against the general movement of

[4]There is, however, no absolute increase in price-volatility associated with herding: As learning continues, in the long-run, volatility settles down and prices react less to individual trades. However, relative to a hypothetical scenario with some intuitive form of naïve trading, the herd-prices are more extreme in the short-term.

prices. Recipients of a monotonic conditional signal distribution are convinced that either extreme outcome has occurred, so no matter what happens they stay on one side of the market.

In the next section we outline the basic setup, the trading equilibrium, the assumptions on signal distributions, and the definition for herding. In Section 3 we revisit Avery and Zemsky's herding example. In Section 4 we discuss which assumptions on MLRP signal structures ensure that herding occurs with positive probability. In Sections 5 and 6 we then show that herding can persist and explain why herding-prices are more extreme. We also briefly discuss Avery and Zemsky's monotonicity concept and how it relates to the MLRP. Most proofs are in the appendix.

2 The Basic Setup and the Trading Equilibrium

Security: There is a single risky asset with a liquidation value V from a set of three potential values $\mathbb{V} = \{V_1, V_2, V_3\} = \{0, \mathcal{V}, 2\mathcal{V}\}$, $\mathcal{V} > 0$. The prior distribution over $\mathbb{V}$ is common knowledge and symmetric around V_2; thus $\mathsf{Pr}(V_1) = \mathsf{Pr}(V_3)$.[5]

Traders: There is a pool of traders consisting of two kinds of agents: *Noise Traders* and *Informed Agents:* At each discrete date t one trader arrives at the market in an exogenous and random sequence. Each trader can only trade once at the point in time at which he arrives. We assume that at each date the entering trader is an informed with probability $\mu > 0$ and a noise trader with probability $1 - \mu$.

The informed agents (also referred to by insiders) are risk neutral and rational. Each receives a private, conditionally i.i.d. signal $S \in \{S_1, S_2, S_3\}$ about V. We assume that the signals are ordered such that $S_1 < S_2 < S_3$.

Noise traders have no information and trade randomly. These traders are not necessarily irrational, but they trade for reasons not included in this model, such as liquidity.[6]

Market Maker: Trade in the market is organised by a market maker who has no private information. He is subject to competition and thus makes zero-expected profits.[7] In every period t, prior to the arrival of a trader, he posts a bid-price p_t^B at which he is willing to buy the security and an ask-price p_t^A at which he is willing to sell the security. Consequently he sets prices in the interval $[V_1, V_3]$.

[5]The symmetry assumption reduce the degrees of freedom and thereby makes it more difficult to establish the possibility of herding.

[6]The existence of noise traders are assumed ($\mu > 0$) to prevent "no-trade" outcomes à la Milgrom-Stokey (1982).

[7]Alternatively, we could also assume a model with many identical market makers setting prices as in Bertrand competition.

Traders' Actions: Each traders can buy or sell *one* unit of the security at prices proposed by the market maker, or he can be inactive. So the set of possible actions is $\mathbb{A} := \{\text{buy}, \text{hold}, \text{sell}\}$. We write $a_t \in \mathbb{A}$ for the action taken in period t by the trader that arrives at that date.

We assume that noise traders trade with equal probability. Therefore, in any period, a noise-trader buy, hold or sale occurs with probability $\gamma = (1-\mu)/3$ each.

Information: The structure of the model is common knowledge among all market participants. The identity of a trader and his signal are private information, but everyone can observe past trades and transaction prices. The history of trades, the sequence of the traders' actions a_t together with the realised transaction prices p_t, is denoted by $H_t = ((a_1, \mathsf{p}_1), \ldots, (a_{t-1}, \mathsf{p}_{t-1}))$ for $t > 1$ and H_1 is the initial history before trade occurred.

2.1 The Trading Equilibrium

The Informed Trader's Optimal Choice: An informed trader enters the market in period t, receives his signal S_t and observes history H_t. We assume the tie-breaking rule that, in the case of indifference, agents always prefer to trade. Therefore, an informed trader's optimal action is (i) to *buy* if he values the security no less than the ask-price: $\mathsf{E}[V|H_t, S_t] \geq \mathsf{p}_t^A$, (ii) to *sell* if he thinks the security is worth no more than the bid price: $\mathsf{p}_t^B \geq \mathsf{E}[V|H_t, S_t]$, and (iii) to *hold* in all other cases.

The Market Maker's Price-Setting: To ensure that the market maker receives zero expected profit the bid and ask prices has be such that at any date t and any publicly available information H_t,

$$\mathsf{p}_t^A = \mathsf{E}[V|a_t = \text{buy at } \mathsf{p}_t^A, H_t], \quad \mathsf{p}_t^B = \mathsf{E}[V|a_t = \text{sell at } \mathsf{p}_t^B, H_t]$$

Insiders are better informed than the market maker. Consequently, if the market maker always sets prices equal to public expectation he makes an expected loss on trades with informed agents. However, if he sets a price above the public expectation he gains on noise traders, as their trades has no information value. Thus, in equilibrium market maker makes profit on trades with noise trader to compensate for losses against insiders. This implies that at any date there is a spread between the bid and ask price; in particular at any date t and for any public information H_t we have

$$\mathsf{p}_t^A > \mathsf{E}[V|H_t] > \quad \mathsf{p}_t^B.$$

Moreover, the spread $\mathsf{p}_t^A - \mathsf{p}_t^B$ increases with μ, the probability of a trader being an insider.

Equilibrium concept. Since the game played by the insiders is one of incomplete information the appropriate equilibrium concept is Perfect Bayesian equilibrium. Therefore, henceforth an equilibrium refers to a profile of actions for each type of insider that constitutes a perfect Bayesian equilibrium of the game. Prices set by the market maker given such action profiles for the insiders are referred to as equilibrium prices.

Price formation in our model is standard:

Remark 1 (Glosten and Milgrom (1985): Standard Results on Price Formation) *Transaction prices form a Martingale process and beliefs converge to the truth.*

The above result describes long-run behaviour of the model. "Convergence results are often overstated. It is certainly more relevant to study how people may be wrong over an extended length of time and how a sudden price change may occur" (Chamley 2004). This is what we study here.

2.2 Properties of the Signal Distribution

We assume that signals are strictly monotonic in the sense of the *monotone likelihood ratio property* (MLRP) (see, for instance, Milgrom (1981)). This means that for any signals $S_l, S_h \in \mathbb{S}$ and any values $V_l, V_h \in \mathbb{V}$ such that $S_l < S_h$ and $V_l < V_h$ we have

$$\mathsf{Pr}(S_l|V_l)\mathsf{Pr}(S_h|V_h) > \mathsf{Pr}(S_l|V_h)\mathsf{Pr}(S_h|V_l).$$

This assumption is standard to models that use informative signals.[8] It is stronger than assuming First Order Stochastic Dominance. We make this very strong (though standard) restriction on the information structure because our objective is to show the possibility of herding (and its consequences) even with a very restrictive condition on the signal distribution.

Before describing some of the (standard) implications of the MLRP, for any signal S we will henceforth employ the following terminology to describe six different types of conditional signal distributions S may have:

$$\begin{aligned}
\text{increasing} &\quad\Leftrightarrow\quad \mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_2) < \mathsf{Pr}(S|V_3)\\
\text{decreasing} &\quad\Leftrightarrow\quad \mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_2) > \mathsf{Pr}(S|V_3)\\
\text{U-shape} &\quad\Leftrightarrow\quad \mathsf{Pr}(S|V_i) > \mathsf{Pr}(S|V_2) \text{ for } i = 1, 3\\
\text{Hill-shape} &\quad\Leftrightarrow\quad \mathsf{Pr}(S|V_i) < \mathsf{Pr}(S|V_2) \text{ for } i = 1, 3
\end{aligned}$$

[8]It is trivially satisfied if there are only two signals, two values and signals are conditionally independent.

$$\text{Negatively biased} \quad \Leftrightarrow \quad \mathsf{Pr}(S|V_1) > \mathsf{Pr}(S|V_3)$$
$$\text{Positively biased} \quad \Leftrightarrow \quad \mathsf{Pr}(S|V_1) < \mathsf{Pr}(S|V_3)$$

We shall also call a signal monotonic if its conditional signal distribution is either increasing or decreasing.

As described in the introduction, with a symmetric prior distribution, a recipient of a hill-shaped signal will tentatively shift weight to the center of the distribution, the recipient of a U-shaped signal will shift weight to the ends. The MLRP allows us to establish the following set of results.

Proposition 1
(a) *Conditional expectations are monotonic in signals: For any* $S_l, S_h \in \mathbb{S}$*, if* $S_l < S_h$ *then* $\mathsf{E}[V|S_l, H_t] < \mathsf{E}[V|S_h, H_t]$ *for any date* t *and any history* H_t*.*
(b) *The conditional signal distribution for* S_1 *is decreasing and the conditional signal distribution for* S_3 *is increasing.*

Proposition 2
In any equilibrium the following holds at any history:
(a) *Informed traders with signal* S_1 *(*S_3*) always sell (buy).*
(b) *The probability of a buy (sale) increases (decreases) in* V *by a positive amount independent of the past. Formally, there exists* $\epsilon > 0$ *such that for every* H_t*, for* $V' > V$*,* $\mathsf{Pr}(buy|V', H_t) - \mathsf{Pr}(buy|V, H_t) > \epsilon$ *and* $\mathsf{Pr}(sale|V, H_t) - \mathsf{Pr}(sale|V', H_t) > \epsilon$*.*

The proofs of the above results are in the Appendix. Proposition 1 (a) implies that investors' conditional expectations are ordered after any history of trade. Since the MLRP implies First Order Stochastic Dominance, it follows that conditional expectations are ordered ex-ante before any trade (see, for instance Milgrom (1981)). This result is simply an extension of this observation to expectations after any history.

Proposition 1 (b) implies that for the lowest (highest) signal, conditional probabilities weakly decrease (increase) in the true liquidation value. However, for the middle signal, no such general rule applies! Conditional probabilities' values can be decreasing, increasing, or they can be hill-shaped or U-shaped with a negative or a positive bias. To see this consider Table 1 which contains six numerical examples of MLRP signal distributions exhibiting all the six conditional signal distributions described above for the middle signal S_2. Each information structure is described by a 3×3 matrices; for each such matrix the MLRP is equivalent to all minors of order 2 being positive. This property holds for all matrices.

Pr($S\|V$)	V_1	V_2	V_3
S_1	$\frac{5}{9}$	$\frac{1}{3}$	$\frac{2}{9}$
S_2	$\frac{6}{18}$	$\frac{4}{18}$	$\frac{3}{18}$
S_3	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{11}{18}$

decreasing for S_2

Pr($S\|V$)	V_1	V_2	V_3
S_1	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{1}{9}$
S_2	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{9}{27}$
S_3	$\frac{7}{27}$	$\frac{1}{3}$	$\frac{5}{9}$

increasing for S_2

Pr($S\|V$)	V_1	V_2	V_3
S_1	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
S_2	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{3}{8}$
S_3	0	$\frac{1}{3}$	$\frac{5}{6}$

hill-shape and positive bias for S_2

Pr($S\|V$)	V_1	V_2	V_3
S_1	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{3}{40}$
S_2	$\frac{20}{120}$	$\frac{40}{120}$	$\frac{11}{120}$
S_3	0	$\frac{1}{3}$	$\frac{5}{6}$

hill-shape and negative bias for S_2

Pr($S\|V$)	V_1	V_2	V_3
S_1	$\frac{31}{100}$	$\frac{1}{5}$	$\frac{1}{100}$
S_2	$\frac{59}{100}$	$\frac{50}{100}$	$\frac{60}{100}$
S_3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{39}{100}$

U-shape and positive bias for S_2

Pr($S\|V$)	V_1	V_2	V_3
S_1	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{50}$
S_2	$\frac{60}{100}$	$\frac{50}{100}$	$\frac{59}{100}$
S_3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{39}{100}$

U-shape and negative bias for S_2

Table 1: **Six Examples of MLRP Signal distributions** For very matrix each entry represents the probability of the row-signal given the true liquidation value given by the column. Therefore, for each matrix the sum of the entries in each column add up to 1. In all the above the signal distributions of S_1 and S_3 are monotonic whereas each matrix exhibits a different kind of signal distributions for S_2.

Proposition 2 (a) establishes that the lowest and highest signal types always take the same action. Therefore, the only agents that might change their behaviour depending on the history of past actions are agents with signal S_2.

Finally note that the conclusions of Proposition 2 (b) hold irrespective of whether players herd.

2.3 Definition and Necessary Conditions for Herding

We adopt the same definition of herding as in AZ.

Definition 1 (Herding) *A trader with signal S engages in* herd-buying *in period t after history H_t if and only if* (H1) $\mathsf{E}[V|S] < \mathsf{p}_1^B$, (H2) $\mathsf{E}[V|S, H_t] > \mathsf{p}_t^A$, (H3) $\mathsf{E}[V|H_t] > \mathsf{E}[V]$. Herd selling *is defined analogously.*

(H1) requires the agent to (strictly) prefer to sell ex-ante, before observing the action of others; (H2) requires the agent to (strictly) prefer to buy, after observing the history; and (H3) requires the public expectation to 'move in the direction' of the herd.

Since $\mathsf{p}_t^A > \mathsf{E}[V|H_t] > \mathsf{p}_t^B$, conditions (H1) and (H2) imply respectively the following necessary conditions for buy-herding

$$\mathsf{E}[V|S] < \mathsf{E}[V] \quad (1)$$

$$\mathsf{E}[V|S, H_t] > \mathsf{E}[V|H_t]. \quad (2)$$

Analogous necessary conditions hold for sell-herding.

Notice that here herding refers to an agent changing his behaviour as a result of observing the behaviour of others.[9] Conditions (H1) and (H2) capture the sense of changing from buy to sell after observing the actions of others. However, an agent may change, for some trading history, from buying to selling (or vice versa) without engaging in herd-behaviour. For example, it may be that he changes, for some trading history, from selling to buying because the market price (public expectations) has fallen. It is important to distinguish herding from such mere 'change of opinion'. Condition (H3) does precisely that by ensuring that the change of action from a buy to a sell is not due to a decline in public expectations.

This change of opinion, in fact is the natural antidote to herd-behaviour. In the literature it is typically referred to as contrarian behavior.[10] To formalize that investors change their opinion to act against the crowd we define contarian-behaviour as follows.

Definition 2 (Contrarian) *A trader with signal S engages in* contrarian-buying *in period t after history H_t if and only if* (C1) $\mathsf{E}[V|S] < \mathsf{p}_1^B$, (C2) $\mathsf{E}[V|S, H_t] > \mathsf{p}_t^A$, (C3) $\mathsf{E}[V|H_t] < \mathsf{E}[V]$. Contrarian selling *is defined analogously.*

The intuition is simple: (C1) requires the agent to (strictly) prefer to sell ex-ante, before observing the action of others; (C2) requires the agent to (strictly) prefer to buy, after observing the history; and (C3) requires the public expectation to have dropped so that after this history a trader who buys acts *against* the general movement of prices.

[9] In the literature there is a debate as to what herding (and informational cascades) exactly entails. For instance, Cipriani and Guarino (2003) and also Smith and Sørensen (2000) define herding as 'action convergence' - agents of the same 'type' take the same action. An informational cascade they describe as a situation where an agent takes the same decision irrespective of his private signal. The aim of AZ's definition is to capture the history-induced switch of opinion in the direction of the crowd. Herding here refers to a particular signal-type, not to all informed agents collectively. In our model the market-maker's zero-profit condition alone precludes action-convergence of all informed traders — it is not possible that all informed investors trade on the same side of the market (See Proposition 2 (1)). In Cipriani and Guarino (2003) action convergence of types is possible because agents differ by type-characteristics other than just signals.

[10] Avery and Zemsky use this term too, but their definition of 'contrarian' includes a feature closely related to their definition of monotonicity; see Section 9.

3 Avery and Zemsky's Event Uncertainty Herding

AZ demonstrate the possibility of herding by employing a special form of asymmetric information labelled *Event Uncertainty*, a concept first developed by Easley and O'Hara (1987). The idea of Event Uncertainty is that ex ante, it is possible that the asset's value has not moved at all. AZ find with Event Uncertainty herding can occur if the information structure of the informed has the following specific form.[11] First, investors know *if* something has happened, i.e. if there was an event which moved the fundamental value of the asset. Second, they receive noisy information about how this event has influenced the liquidation value. Formally,

$$\begin{aligned} \mathsf{Pr}(S_1|V_1) &= \mathsf{Pr}(S_3|V_3) = q > .5, \quad \mathsf{Pr}(S_3|V_1) = \mathsf{Pr}(S_1|V_3) = 1-q, \\ \mathsf{Pr}(S_2|V_3) &= \mathsf{Pr}(S_2|V_1) = 0, \quad \mathsf{Pr}(S_2|V_2) = 1. \end{aligned} \tag{3}$$

There are two points to note concerning AZ's Event Uncertainty structure. First, it clearly is not MLRP-monotonic: consider signals S_3, S_2 and V_1, V_2; then

$$\mathsf{Pr}(S_3|V_2)\mathsf{Pr}(S_2|V_1) = 0 \cdot 0 = 0 < 1-q = 1 \cdot (1-q) = \mathsf{Pr}(S_2|V_2)\mathsf{Pr}(S_3|V_1),$$

violates the MLRP. Second the insiders' information is always better than the market maker's in the sense that signals are informative (but not fully revealing). Knowing that some liquidation value has not occurred is an additional piece of information that causes investors' partitions of the set of liquidation values to be finer than the market maker's. Avery and Zemsky interpret this as a different 'dimension' of uncertainty and attribute their herding result to this property. Our analysis in the next subsection, in fact, suggests that it is not the 'dimension' but rather the general shape of the conditional signal distribution that determines if herding is possible. Even if the insider believes that with small probability the "no-information" event V_2 has happened he may still herd.

We now state the herding result of AZ with Event Uncertainty.

Proposition 3 (Avery and Zemsky (1998), Proposition 5)
Suppose that the information structure satisfies (3). Then herding occurs with a positive probability at some finite history.

The proof of this result with Event Uncertainty is particularly simple and compelling. It suffices to find a finite history that satisfies the following: first, there is a series of holds for the first n periods, which leaves the insider's beliefs unchanged, but which causes the

[11] In what follows we will use the Event Uncertainty to indicate this information structure.

marketmaker's conditional distribution to place a larger weight on the middle value V_2. Then there is m periods of buys which increases the insiders' expected value. Now if both n and n/m are sufficiently large then the market maker's conditional distribution after this history places a large weight on the middle value V_2 (this follows from n/m being large) whereas the insider's conditional distribution after this history place a large weight on V_3 (this follows from n being large). Thus after such a history S_1 traders' expectation rises above the ask-price. Since S_1 types initially sell it then follows that they will change their action in the direction of the crowd and herd after this history. (See AZ for details of the proof).

4 Herding with MLRP Signal Structures

AZ's Event Uncertainty information structure allows herding. So is there a more general lesson to learn from the underlying signal distribution? In their proof the buy-herding investor had the lowest signal S_1. An interesting feature of this agent's signal distribution is that it is U-shaped: $\mathsf{Pr}(S_1|V_i) > \mathsf{Pr}(S_1|V_2)$ for $i = 1, 3$.

As we noted before, AZ's Event Uncertainty information structure is not consistent with MLRP. On the other hand, by Proposition 2 (a) and Proposition 1 (b), for any information structure that satisfies MLRP we have respectively

• the only possible herding candidate is an investor with middle signal S_2 (S_1 traders always sell and S_3 traders always buy);

• the only type of investors that could have a non-monotonic (and in particular U-shaped) conditional signal distribution has signal S_2 (S_1 and S_3 must have a monotonic conditional signal distributions). See Table 1 for examples.

The above two observations allows us to establish that herding is possible for insiders with signal S_2. More specifically, we show below that, with MLRP signals, the following two conditions are necessary and sufficient conditions for herding (by S_2 types):

• U-shaped signal distribution for signal S_2;

• 'enough' noise.

In addition, we show that, depending on the relative values of $\mathsf{Pr}(S_2|V_1)$ and $\mathsf{Pr}(S_2|V_3)$, either buy-herding is possible or sell-herding but not both. This is because for the informed investor with signal S_2 to buy-herd he must have a negative opinion prior to the beginning of trading - condition (1). Since the prior is symmetric this is equivalent to S_2's conditional signal distribution being negatively biased:

$$\mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_3) \tag{4}$$

Analogously, for the case of S_2 to sell-herd we need the additional condition that $\mathsf{E}[V|S_2] > \mathsf{E}[V]$; this is equivalent to S_2's conditional signal distribution being positively biased:

$$\mathsf{Pr}(S_2|V_1) < \mathsf{Pr}(S_2|V_3) \tag{5}$$

Next we present our characterisation results for herding formally and then explain these conditions in greater detail.

Existence of Herding

First define

$$\begin{aligned} \kappa_b &:= \frac{\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)}{\rho_{23}^{23}}, \quad \kappa_s := \frac{\mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_2)}{\rho_{12}^{12}} \\ \theta_b &:= \frac{\mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_3)}{\mathsf{Pr}(V_2)(\rho_{12}^{12} + \rho_{12}^{23}) + (1 - \mathsf{Pr}(V_2))\rho_{12}^{13}}, \quad \theta_s := \frac{\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)}{\mathsf{Pr}(V_2)(\rho_{23}^{12} + \rho_{23}^{23}) + (1 - \mathsf{Pr}(V_2))\rho_{23}^{13}} \end{aligned}$$

where

$$\rho_{ij}^{kl} = \mathsf{Pr}(S_i|V_k)\mathsf{Pr}(S_j|V_l) - \mathsf{Pr}(S_j|V_k)\mathsf{Pr}(S_i|V_l).$$

Proposition 4 (Herding with MLRP Signals)
(a) The investor with signal S_2 buy-herds with positive probability if and only if $0 < \mu < \mu_b$ *where*

$$\mu_b \;=\; \min\{\mu_b^{in}, \mu_b^{ch}\}, \quad \mu_b^{in} = \theta_b/(\theta_b + 3), \quad \mu_b^{ch} = \kappa_b/(\kappa_b + 3).$$

(b) The investor with signal S_2 buy-herds with positive probability if and only if $0 < \mu < \mu_s$ *where*

$$\mu_s \;=\; \min\{\mu_s^{in}, \mu_s^{ch}\}; \quad \mu_s^{in} = \theta_s/(\theta_s + 3), \quad \mu_s^{ch} = \kappa_s/(\kappa_s + 3).$$

We show the result directly, by exploiting properties of the MLRP. As part of the proof we show that after any history of length t, $\mathsf{E}[V|S_i, H_t] - \mathsf{p}_t^A$ has the same sign as

$$[\beta_2^t\mathsf{Pr}(S_i|V_3) - \beta_3^t\mathsf{Pr}(S_i|V_2)] + \frac{q_1^t}{q_3^t}[\beta_1^t\mathsf{Pr}(S_i|V_2) - \beta_2^t\mathsf{Pr}(S_i|V_1)] + \frac{2q_1^t}{q_2^t}[\beta_1^t\mathsf{Pr}(S_i|V_3) - \beta_3^t\mathsf{Pr}(S_i|V_1)] \tag{6}$$

and $\mathsf{E}[V|S_i, H_t] - \mathsf{p}_t^B$ has the same sign as

$$\frac{q_3^t}{q_1^t}[\sigma_2^t\mathsf{Pr}(S_i|V_3) - \sigma_3^t\mathsf{Pr}(S_i|V_2)] + [\sigma_1^t\mathsf{Pr}(S_i|V_2) - \sigma_2^t\mathsf{Pr}(S_i|V_1)] + \frac{2q_3^t}{q_2^t}[\sigma_1^t\mathsf{Pr}(S_i|V_3) - \sigma_3^t\mathsf{Pr}(S_i|V_1)] \tag{7}$$

where $\beta_i^t = \mathsf{Pr}(\text{buy}|H_t)$, $\sigma_i^t = \mathsf{Pr}(\text{sale}|H_t)$ and $q_i^t = \mathsf{Pr}(V_i|H_t)$ denote respectively the conditional probability of a buy, the conditional probability of a sale and the conditional probability of state V_i at date t. Since S_1-types always sell and S_3-types always buy it follows that for S_1-types (6 and (7) are always negative, and for S_3-types they are always positive, irrespective of the trading histories.

This leaves the S_2-types as herding candidates. To have buy-herding, we must first ensure that (7) is negative at the initial history (and therefore, S_2-types sell at the initial history). This is true if and only if $0 < \mu < \mu_b^{in}$. The intuition here is that we need a sufficient amount of noise traders (more than $1-\mu_b^{in}$) to ensure that an S_2-signal investor is initially on the sale-side of the market because the more noise there is, the smaller the difference between the bid-price and the ask-price. For $\mu < \mu_b^{in}$ the bid-ask spread is sufficiently small so that it is worth for S_2-types to sell at the initial history; when $\mu > \mu_b^{in}$ the spread is too large and therefore there is not enough inducement for S_2-types to sell at the initial history.

Next to have herding after a finite history, it must be true that some terms in (6) are positive and some negative. Requiring $0 < \mu < \mu_b^{ch}$ is necessary and sufficient for the first term in (6) to be positive. Moreover, if there is sufficient evidence in favour of high values (i.e. sufficiently many more 'buys' than 'sales') then the last two terms in (6) vanish. This is because, by part (*b*) of Proposition 2, the posterior probability q_1^t will be sufficiently small relative to q_2^t and q_3^t. Hence the expression in (6) is indeed positive and S_2 types change their behaviour at such a history if $0 < \mu < \mu_b^{ch}$. The reverse direction works analogously.

Therefore the two conditions $0 < \mu < \mu_b^{in}$ and $0 < \mu < \mu_b^{ch}$ (initially (*in*) buy and later change (*ch*) together are necessary and sufficient conditions for buy-herding.

Clearly, to ensure that the above characterisation result is not vacuous μ_b and μ_s have to be positive. However, notice that by the MLRP we have

$$\begin{aligned} \kappa_b &> 0 \Leftrightarrow \mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2) > 0 \\ \kappa_s &> 0 \Leftrightarrow \mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_2) > 0 \\ \theta_b &> 0 \Leftrightarrow \mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_3) > 0 \\ \theta_s &> 0 \Leftrightarrow \mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_1) > 0 \end{aligned}$$

Therefore, $\mu_b > 0$ ($\mu_s > 0$) if and only if $\mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_2)$. Also $\mu_s > 0$ if and only if $\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_2)$. Thus we have the following corollary to the above result.

Corollary (Necessary and Sufficient Conditions Revisited)
(*a*) *There exists $\bar{\mu} > 0$ such that for any $0 < \mu \leq \bar{\mu}$ there is a positive probability of*

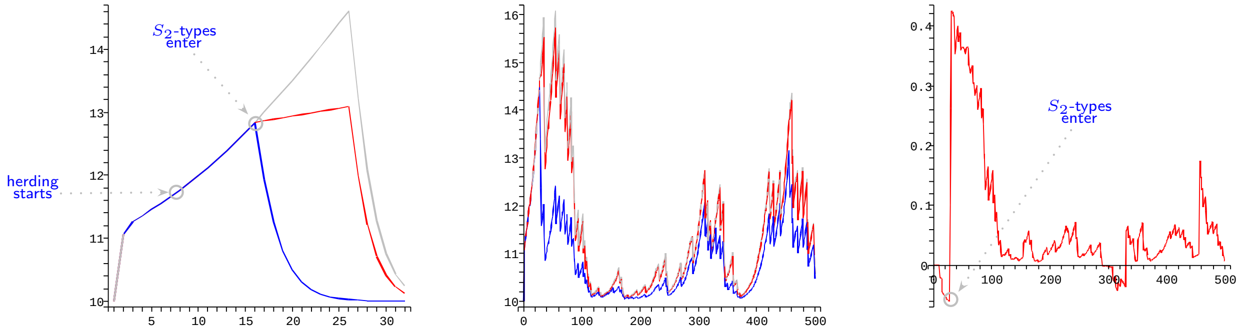


Figure 1: **Illustrations of the Sensitivity in Prices Paths with and without Herding.**

buy-herding if and only if S_2's signal distribution is negatively biased and U-shaped:

$$\mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_2).$$

(b) There exists $\bar{\mu} > 0$ such that for any $0 < \mu \leq \bar{\mu}$ there is a positive probability of sell-herding if and only if S_2's signal distribution is positively biased and U-shaped

$$\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_2).$$

Importantly, 'hill-shape' or 'monotonicity' of one signal's distribution cannot be interpreted as (global) 'signal-monotonicity' — these properties say nothing about the relation between signals. When considering the impact that the U-shaped signal has on the recipient's posterior (he shifts weight to the extreme values), our result is very intuitive: it is the group of somewhat confused traders, those who received 'mixed'-information who are prone to run with the herd.

Coming back once more to Avery and Zemsky's herding, one should note that their Event Uncertainty signal structure has two signals with a U-shaped distribution: signals S_1 and S_3. Since this structure is not MLRP, one can no longer argue that for sufficiently many more buys than sales the expression in (6) is positive. However, the intuition is similar: first, sufficiently many holds can make the third term in (6) arbitrarily small. Sufficiently many buys do the rest by making the second term arbitrarily small, eventually triggering herding. It is immediately clear that our reasoning can also be applied to small perturbations of Avery and Zemsky's Event Uncertainty signal structure (with non-zero conditional signal distribution). Since such a perturbed information structure would no longer be multidimensional it follows that multidimensionality is not the issue in inducing herd behaviour.

5 Persistence of Herding

In Avery and Zemsky price movements during herding under Event Uncertainty are strictly limited: For informed investors, trades do not convey information, thus their expectation does not move. To break buy-herding (sell-herding), it suffices that prices rise above (fall below) the (constant) expectation of S_1 (S_3) types, and this is generally a very small movement. In fact, in AZ the required price movement during any herding vanishes in the limit as $\mu \to 0$ and $q \to 1/2$ (as the informativeness of the signals of the informed agents disappears); see Proposition 8 in AZ.

In our setting, prices may move significantly during herding. In fact, if buying persists and there are no sales, buy-herding will not stop (because of noise trading this is, of course, a zero probability event). Under buy-herding, further buys will increase the herders' expectation stronger than the marker maker's and thus the herd is not broken. The same reasoning holds for sell-herding. Once herding starts, buying will also get more likely as now S_3- and S_2-types buy. Likewise, the herd is quite robust – breaking it gets more difficult the more herd-buys there are.[12]

The intuition for the result is as follows: By part (b) of Proposition 2 the probability of a buy is strictly increasing for higher liquidation values, thus both market maker and insider consider higher values more likely during buy-herding. However, once the conditions for herding hold, the S_2-investors are on the favourable side of the market, and thus update higher values faster than the market maker. If buy-herding starts then the second and the third terms in (6) are small relative to the first term. By Proposition 2 (b), for $V_i < V_j$, $\Pr(V_i|H_t)/\Pr(V_j|H_t)$ declines in buys. Therefore, more buys simply make the second and the third terms in (6) more insignificant and thus they will remain 'sufficiently small' relative to the first term. This causes herding to persist. Further, buy-herding is persistent as long as there are not 'too many' sales. Indeed, once herding starts, sales work in the same way as buys did prior to herding — if there are sufficiently many sales so that the second and third terms in (6) get large, then the S_2-insider's expectation drops below the ask-price.

Moreover, herding can start when prices are close to the middle value, V_2. As the prior probability for an extreme value goes to zero, $\Pr(V_2) \to 1$, the price movement needed to trigger herding becomes small. Moreover, as $\Pr(V_2) \to 1$, the minimum number of necessary same-direction trades becomes independent of the prior on V. As a result, buy-herding (sell-herding) can start at a price close to V_2. Figure 2 plots simulated

[12] This is also in contrast to standard herding model as in Bikchandani et al (1992). In the latter herding is ever fragile: since all players ignore their information a single contrarian action against the herd results in a collapse of the herd. Not so here, as all actions are informative.

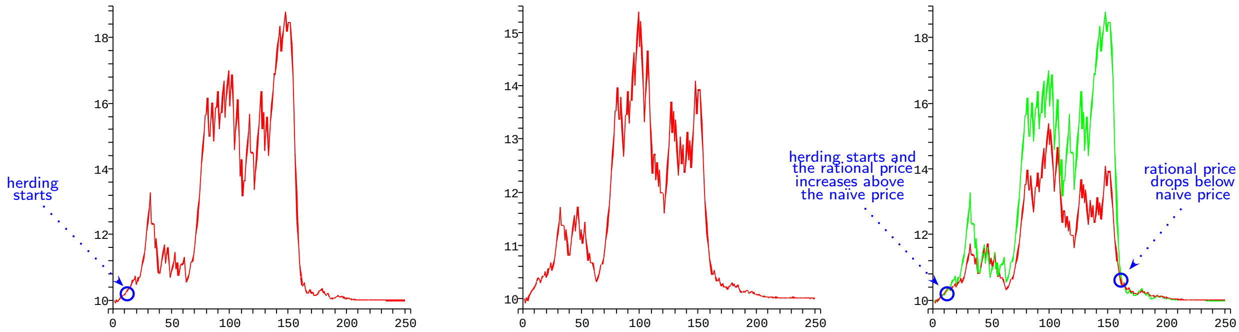


Figure 2: **Simulated Transaction Prices.** The left panel displays a simulation of transaction prices when traders behave rationally (and thus herd). As can be seen, herding starts for low prices, and prices during herding can move up substantially ($V_3 = 20,\ V_2 = 10$). The middle panel plots transaction prices for the same sequence of traders, but for "naïve" investors, i.e. the S_2 investors merely follow their prior expectation and ignore all information in the trading history. The right panel combines both scenarios and highlights the point where the naïve prices exceed the rational price (note: S_2-types are still herding at this point). Details of the data are available from the authors upon request. The underlying signal distribution is listed in Appendix B.

transaction prices that illustrate this point: Herding starts for prices near V_2, and during herding prices rise substantially.

Proposition 5 (Persistence of Herding and the Range of Herd-Prices)
(*a*) *Suppose that the signal distribution for signal* S_2 *is negatively biased and U-shaped and that after history* H_t *there is buy-herding. Then for any* $\epsilon > 0$, *there exists history* $H_{t+\tau}$ *so that there is buy-herding at every* $\tilde{t} \in [t, t+\tau]$ *and transaction price* $\mathsf{p}_{t+\tau}$ *exceeds* $V_3 - \epsilon$.
(*b*) *Suppose that the signal distribution for signal* S_2 *is positively biased and U-shaped and that after history* H_t *there is sell-herding. Then for any* $\epsilon > 0$, *there exists history* $H_{t+\tau}$ *so that there is sell-herding at every* $\tilde{t} \in [t, t+\tau]$ *and transaction-price* $\mathsf{p}_{t+\tau}$ *is below* $V_1 + \epsilon$.
(*c*) *Assume* $\mathsf{Pr}(V_2) = 1 - \frac{1}{n}$. *Then for every* $\epsilon > 0$ *there exists an* $n < \infty$ *so that the smallest price with herd-buying* p^* *is in* $(V_2, V_2 + \epsilon)$.

6 Price Movements during Herding

Two questions arise naturally when buy-herding starts: First, will buys move prices less with herding than when herding/social learning is not allowed? And second, will sales move prices more with herding than when no herding/social learning is allowed? In what follows we focus on price-impacts for buy-herding; sale-herding effects are analogous.

To answer this question we compare bid- and ask-prices in a buy-herding situation (we refer to this as the *rational* case) (sell-herding is analogous) with prices in a hypothetical economy (called *naïve*) in which

• at each date t, S_2-agents are naïve and unable (unwilling) to interpret the public information; they therefore buy (sell) if their expected value conditional on their private information $\mathsf{E}[V|S_2]$ exceeds the ask price (is less than the bid price) and hold otherwise;

• the market maker sets prices as before taking into account that the S_2-types' strategies are indeed naïve.

Casual intuition suggests that once buy-herding is possible, buys in a rational world should move prices less and sales more than in the naïve world. This is because, loosely,

• a 'buy' carries less information when agents are rational and may buy-herd (both S_2 and S_3 buy) than when they do not (only S_3 buys),

• a 'sale' is a stronger negative signal in the rational buy-herd case (only S_1 sells) than in the naïve (S_1 and may be S_2 sell) case.

While the intuition for sales is accurate, the intuition for buys is misleading: when buy-herding starts prices move stronger in *both* directions. Again, the reason lies in the U-shape of the S_2-signal distribution. At any history H_t at which buy-herding starts a buy in the naïve world reveals that the buyer is either an S_3-type or a noise trader whereas in the rational world a buy reveals that the buyer is an S_2, S_3 or a noise trader. Prices set by the market maker differ because of the information inferred from signal S_2: when there is herding, signal S_2 alone suggests a large weight on V_3 and little on other values because

• S_2 has a U-shaped signal distribution and therefore given S_2 the weight on V_2 is small whereas weight on V_1 *and* V_3 is large;

• there has been a sufficiently large number of no-herd buys (buy-herding has started); this indicates a large number of S_3 signals and thus a low weight on V_1 at H_t.

To compare the naïve and the rational case, we introduce the following notation for the naïve case: let $\mathsf{E}_n[V|H_t], \mathsf{p}^A_{t,n}, \mathsf{p}^B_{t,n}, \beta_{i,n}, \sigma_{i,n}$ be respectively the public (market) expectation, the ask-price, the bid-price, the probability of a buy in state i and the probability of a sale in state i. From now on, assume that $0 < \mu < \mu_b$ so that buy-herding is possible. Next there is a history $H_t = (a_1 \ldots, a_{r+b+s})$, where

(I1) for any truncation $H_\tau = (a_1, \ldots, a_\tau)$ a buy-herd is possible if and only if $\tau \geq r$,

(I2) the path $(a_{r+1}, \ldots, a_{r+b+s})$ consists of b buys and s sells,

(I3) posteriors q_i^r are identical for the rational and the naïve case.

For the following proposition we further restrict the amount of noise. Define

$$\kappa_{hb} = \frac{\mathsf{Pr}(S_2|V_1) - \mathsf{Pr}(S_2|V_3)}{\rho_{12}^{13}}, \tag{8}$$

$$\kappa_p = \frac{\mathsf{Pr}(S_3|V_2) - \mathsf{Pr}(S_3|V_1) + \mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_3|V_1)}{\mathsf{Pr}(S_3|V_3)(\mathsf{Pr}(S_3|V_2) - \mathsf{Pr}(S_3|V_1)) + \mathsf{Pr}(S_2|V_3)\mathsf{Pr}(S_3|V_2) - \mathsf{Pr}(S_2|V_1)\mathsf{Pr}(S_3|V_3)}. \quad (9)$$

Proposition 6 (The Impact of Herding on Prices)
Consider any history $H_t = (a_1, \ldots, a_{r+b+s})$ *that satisfies* (I1)–I3
(a) *If* $s = 0$ *then for all* b $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] > 0$.
(b) *Assume* $\mu > \kappa_{hb}/(3 + \kappa_{hb})$ *and let* $b = 0$. *If after* s *sales,* $\mathsf{E}[V|S_2, H_t] > \mathsf{p}_t^B$, *then* $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] < 0$.
(c) *If either*
(i) $\mathsf{Pr}(S_3|V_2) - \mathsf{Pr}(S_3|V_1) + \mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_3|V_1) < 0$ *or*
(ii) $\mu < \kappa_p/(\kappa_p + 3)$,
then for any s *there exists* $\bar{b}$ *such that for all* $b > \bar{b}$ $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] < 0$.
(d) *The ex ante probability of a buy is higher during buy-herding than outside herding.*

Part (a) shows that the rational ask-price is and remains above the naïve ask price as long as buying persists. Part (b) shows that as long as the rational S_2-type remains in herding-mode, the rational bid-price is and remains below the naïve bid-price when people keep *selling*.

Part (c) is easiest to understand when reading it backwards: After many herd-buys, both the naïve and the herding price converge to the highest value and thus they are close. Then after a number of sales, the rational ask-price can drop below the naïve ask-price. [13] Part (c) thus shows that for herd-prices are more sensitive than naïve prices if they are sufficiently close. Conditions (i) and (ii) ensure that in the rational case, sales have a sufficiently strong effect on the posterior of V_1; (i) is sufficient for (ii).

Figure 2 illustrates the proposition: The left panel displays rational prices, the middle panel displays naïve prices, and the right panel plots both simultaneously.

7 Contrarians

Contrarian behaviour is a simple side-product of our herding analysis. As with herding, to identify such trading behavior, we need to ensure a minimum level of noise. The relevant

[13] Exactly when herding starts, the rational ask-price is above the naïve ask price, and the rational bid-price is below the naïve bid price. But after, say, some buys, it is difficult to determine whether the rational bid is still below naïve bid — simply because the posterior expectations change differently. But once expectations are sufficiently close, which is ensured after many herd-buys, the "more extreme" movements statement holds again.

threshold values here are given by

$$\kappa_b^{\text{con}} := \frac{\mathsf{Pr}(S_2|V_2) - \mathsf{Pr}(S_2|V_1)}{\rho_{23}^{12}}, \quad \kappa_s^{\text{con}} := \frac{\mathsf{Pr}(S_2|V_2) - \mathsf{Pr}(S_2|V_3)}{\rho_{12}^{23}}.$$

Corollary (Necessary and Sufficient Conditions for Contrarian Behaviour)
(a) The investor with signal S_2 is a buy-contrarian with positive probability if and only if (i) $0 < \mu \leq \mu_b^c$ where

$$\mu_b^c = \min\{\mu_b^{con}, \mu_b^{in}\}, \ \textit{with} \ \mu_b^{con} = \kappa_b^{con}/(\kappa_b^{con} + 3),$$

and (ii) the S_2-signal distribution is negatively biased and hill-shaped

$$\mathsf{Pr}(S_2|V_2) > \mathsf{Pr}(S_2|V_1) > \mathsf{Pr}(S_2|V_3)$$

(b) The investor with signal S_2 is a sell-contrarian with positive probability if and only if (i) $0 < \mu \leq \mu_s^c$ where

$$\mu_s^c = \min\{\mu_s^{con}, \theta_s/(\theta_s + 3)\}, \ \textit{with} \ \mu_s^{con} = \kappa_s^{con}/(\kappa_s^{con} + 3),$$

and (ii) the S_2-signal distribution is positively biased and hill-shaped

$$\mathsf{Pr}(S_2|V_2) > \mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1).$$

The result can be straightforwardly obtained when rescaling (6) with q_3^t/q_1^t

$$\frac{q_3^t}{q_1^t}[\beta_2^t\mathsf{Pr}(S_i|V_3) - \beta_3^t\mathsf{Pr}(S_i|V_2)] + [\beta_1^t\mathsf{Pr}(S_i|V_2) - \beta_2^t\mathsf{Pr}(S_i|V_1)] + \frac{2q_3^t}{q_2^t}[\beta_1^t\mathsf{Pr}(S_i|V_3) - \beta_3^t\mathsf{Pr}(S_i|V_1)] \quad (10)$$

And for sales we can rescale (7) with q_1^t/q_3^t

$$[\sigma_2^t\mathsf{Pr}(S_i|V_3) - \sigma_3^t\mathsf{Pr}(S_i|V_2)] + \frac{q_1^t}{q_3^t}[\sigma_1^t\mathsf{Pr}(S_i|V_2) - \sigma_2^t\mathsf{Pr}(S_i|V_1)] + \frac{2q_1^t}{q_2^t}[\sigma_1^t\mathsf{Pr}(S_i|V_3) - \sigma_3^t\mathsf{Pr}(S_i|V_1)] \quad (11)$$

For sufficiently many more sales than buys, the first and last terms in (10) will vanish. The condition on μ stated in (a) ensures that the second term is positive. On a broader scale, an investor with a hill-shaped S_2 signal increases his posterior on V_2 — he is more certain that the true value is V_2. Thus if prices fall he would buy as the asset seems relatively undervalued to him, making this investor a contrarian.

8 History Dependence and the Probability of Herding

Simple History Dependence. In the long run, prices will converge to the true value.[14] If model primitives do not allow herding or contrarian behavior, then the order of trades and traders is irrelevant.

This is not true if there can be herding. Consider the following numerical example of a herding-MLRP signal structure[15]

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{40}{49}$	$\frac{4}{49}$	0
S_2	$\frac{9}{49}$	$\frac{9}{490}$	$\frac{243}{12250}$
S_3	0	$\frac{9}{10}$	$\frac{12007}{12250}$

$\mu = \frac{1209}{1600}$,
$\mathbb{V} = (0, 10, 20)$, and
$\mathsf{Pr}(V) = (1/6, 2/3, 1/6)$.

For illustrative purposes, assume that the first fifteen traders are all informed, and let each signal be held by five investors.

SERIES 1: S_1–S_2–S_3 OR S_1–S_3–S_2. The S_1 traders move first; all sell and thus the price drops. Irrespective of the S_3-types' buys, the S_2-types still sell. So after these 15 trades the public expectation will drop from 10 to .15.

SERIES 2: S_3–S_2–S_1. The S_3 traders move first and buy. Then the S_2 types will be subject to herding and buy, too. The public expectation now rises to about 13.5. The five S_1-type sell, and then the public expectation drops to 10.31.

These two series illustrate how the arrival order of traders matters: since there are S_2-types who trade, this type's change in trading-mode (from selling to buying) directly affects prices. In the next two series, the same number and type of trades occurs, but in different orders.

SERIES 3: 20 BUYS – 20 SALES. After 20 buys, the public expectation is 15.36, after 20 subsequent sales it is 3.12.

SERIES 4: 20 SALES – 20 BUYS. After 20 sales, the public expectation is 1.16×10^{-13}, after 20 subsequent buys it is 10.0064.

In summary, the order of trades influences the frequency of future trades because the type S_2 changes modes, and it influences prices because the S_2-type's different trading-mode has to be accounted for. Although in the long run there is convergence, in the short run such fluctuations matter.

[14] This is a fairly general result and can be shown in many different ways, for instance by employing O'Hara (1997)'s[p. 84-86] textbook-tools such as the log-likelihoodratio method. We omit a proof.

[15] We chose the numbers so that there can be herding after a small number of trades.

Price Sensitivity. To further elaborate on the price sensitivity induced by herding, consider the price paths described in Figure 1: In the left panel, we plot the price path with $\mu = \mu_b \pm \epsilon$ for rational players and also for the naïve investors,[16] where parameters are chosen so that $\mu_b \equiv \mu_b^{ch}$ — this ensures that a small reduction in μ from $\mu_b + \epsilon$ to $\mu_b - \epsilon$ can trigger herding. For $\mu = \mu_b + \epsilon$, there is no herding, thus if S_2-type traders enter, they either sell or hold. If, however, $\mu = \mu_b - \epsilon$, S_2-type traders buy, and price paths are stronger upward-biased. In the middle panel we plot a price-series for both rational and naïve traders with $\mu_b + \epsilon$, and, for the same history of trader-identities, the rational prices for $\mu_b - \epsilon$ (so that herding is possible). In the right panel we plot the difference of the two rational price-series. Since there is more noise in the herding prone series, initially, the no-herd-series' price is above the rational herd series' price (the right panel in Figure 1 shows this). Once herding starts, however (here after 8 trades), and once an S_2 type enters, this relation flips.

The Probability of Herding. The shortest sequence of trades that leads to buy-herding is one with only buys. To assess the probability of herding, we simulate for given set of parameters (i) how many same direction trades are needed for herding, and (ii) how likely this sequence is. As the amount of informed trading increases from 0 to μ_b, there are two opposing effects: First, as noise decreases, the positive term in (6) (the first term) becomes smaller, so it gets more difficult to obtain herding. Second, the second and third terms in (6), the negative terms, decline faster in μ for every buy: observe that

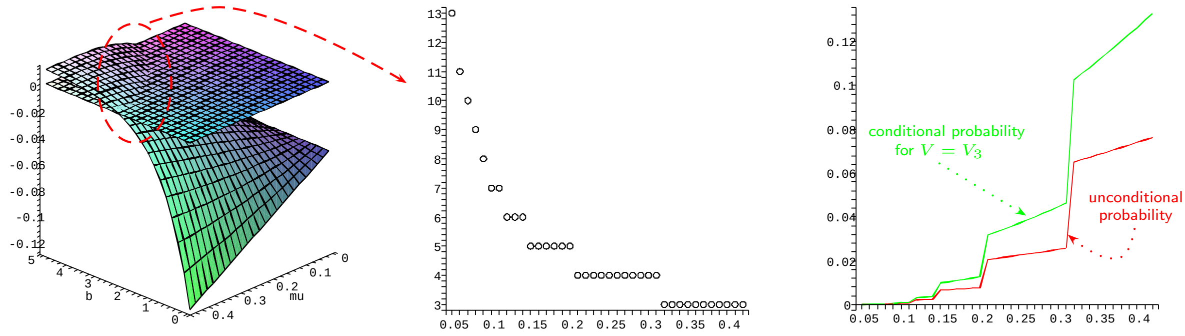
$$\frac{\beta_1}{\beta_3} = \frac{\mu \mathsf{Pr}(S_3|V_1) + \gamma}{\mu \mathsf{Pr}(S_3|V_3) + \gamma}, \quad \text{and thus} \quad \frac{\partial}{\partial \mu}\frac{\beta_1}{\beta_3} = (\mathsf{Pr}(S_3|V_1) - \mathsf{Pr}(S_3|V_3))/{\beta_3}^2 < 0,$$

where the inequality is strict due to the strict MLRP. Similarly, $\partial/\partial\mu(\beta_1/\beta_2) < 0$. As noise decreases, each trade gets more informative.

While an analytical result seems out of reach as there are too many parameters that determine the result, the second effect dominates in all numerical examples that we computed. Thus as noise trading declines (μ increases to μ_b) it takes *less* same-direction trades to trigger herding. Figure 3 plots the minimum number of such consecutive time-zero buys needed to trigger herding.

Moreover, as the amount of noise decreases, ex ante it gets more likely that these consecutive buy-trades occur. Figure 3's right panel illustrates these probabilities.

[16] For naïve the differences in prices for the two levels of μ are negligible.



Equation (6) Minimum b given μ Probability of b given μ

Figure 3: **Trades needed for Herding and Herding Probabilities.** The left panel plots equation (6) for a given signal distribution as a function of μ, where μ ranges from (almost) 0 to the maximally possible μ_b, and of no-herd buys. Whenever the bend curve lies above the 0-surface, there is herding. The middle panel computes the minimum integer number of no-herd buys that would trigger herding. The right panel computes from the middle panel the probability of having exactly the threshold number of buys at the beginning of trade, depending on the underlying μ. As can be seen, conditional on the true value being V_3, the probability of this herding history increases in μ, and it is larger than the unconditional probability of such histories. The underlying signal distribution is listed in Appendix B.

9 Avery and Zemsky's Signal Monotonicity and the MLRP

Avery and Zemsky argue that it is the Event Uncertainty information structure's inherent non-monotonicity that triggers herding. They use the following definition of signal monotonicity,

Definition 3 (Monotonicity)
A signal S is monotonic *if there exists a function $w(S)$ such that for all histories H_t, $\mathsf{E}[V|S, H_t]$ is always weakly between $\mathsf{E}[V|H_t]$ and $w(S)$.*

Avery and Zemsky show that this signal monotonicity precludes herding. This is not surprising, however, as this definition is written in such a way that it almost immediately rules out herding by definition. The definition does not establish conditions on the primitives of the signal distribution, but rather employs a requirement on all trading histories. The following proposition clarifies how that AZ's monotonicity definition is implied by a conditional signal distribution that is monotonic in values.

Proposition 7 (Equivalence of the Monotonicity Concepts)
Assume that the signal distribution satisfies the MLRP. If $\mathsf{Pr}(S|V)$ is monotonic in V then signal S satisfies Definition 3.

Monotonicity of $\mathsf{Pr}(S|V)$ in V immediately implies that (6) and (7) always have a constant sign. One can straightforwardly construct examples with non-MLRP distributions that do not satisfy the MLRP.

10 Conclusion

We provide a necessary and sufficient condition for positive probability herding in a world with three liquidation values and monotonic (MLRP) signal distributions. For herding to occur trades must be sufficiently uninformative, but also, the middle types' conditional signal distribution must be 'U-shaped' in liquidation values. The intriguing part about the distribution requirement is the effect that such a signal distribution has on the agent's posterior: While their information is valuable and informative, in their posteriors these agents shift weight from the middle to the extreme values. Thus one can say that herding-prone investors have been *confused* by their signal. By the same token, contrarian investors are highly convinced of their information.

Notably, while the required signal distribution properties are special, they are still admissible under the MLRP. The MLRP itself, however, is a tight corset on signal distributions, and there is no reason to believe that MLRP-signals are prevalent in the real world — researchers often use them for their convenience when solving models analytically. The amount of herding in our model is restricted by the likelihood of middle-signal types, and the MLRP places tight bounds on this likelihood. But in reality, when there are no MLPR signals, it seems even more likely that there is is a large fraction of somewhat informed yet "confused" investors who fit exactly into the herding-prone signal-category!

Herding in our model by its definition does not imply that all traders act alike, and thus does not match the intuition suggested in flashy newspaper headlines. Rather, it signifies a substantial shift in "sentiment", which involves an accelerating rate of same-direction trades (e.g. 'buys beget more buys').

For there to be a flush of buys, someone has to sell. In sequential trading models, all trades go through the market maker who is compelled to trade. This merely is by design — and the same design also forbids that all traders act alike (even in AZ this is true). Realistically, even when markets crash, market makers do not pick up all sales immediately, there will be others who trade against the stream. And it is not conclusive to assume that these are all noise traders. In other words, while our herding does not yield uniform behavior, it does capture short-term swings in sentiment.

To summarize, our model clarifies that rational herding requires confusion, while contrarian behavior needs conviction.

A Omitted Proofs

Proof of Proposition 1

(a) By standard results on MLRP and stochastic dominance (see, for instance, Hirshleifer and Riley (1992), [p. 106]) it must be that $\mathsf{E}[V|S_l] < \mathsf{E}[V|S_H]$. Next consider any history H_t. First, note that

$$\mathsf{Pr}(V|H_t, S) = \mathsf{Pr}(V|S)\mathsf{Pr}(H_t|V) / \sum_{V' \in \mathbb{V}} \mathsf{Pr}(V'|S)\mathsf{Pr}(H_t|V').$$

Then for $V_h > V_l$ and $S_h > S_l$ the MLRP condition after history H_t,

$$\mathsf{Pr}(S_l|H_t, V_l)\mathsf{Pr}(S_h|H_t, V_h) > \mathsf{Pr}(S_h|H_t, V_l)\mathsf{Pr}(S_l|H_t, V_h),$$

still holds by the following manipulations

$$\begin{aligned}
\mathsf{Pr}(V_l|H_t, S_l)\mathsf{Pr}(V_h|H_t, S_h) &> \mathsf{Pr}(V_h|H_t, S_l)\mathsf{Pr}(V_l|H_t, S_h) \\
\Leftrightarrow \frac{\mathsf{Pr}(V_l|S_l)\mathsf{Pr}(H_t|V_l)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_l)\mathsf{Pr}(H_t|V)} \frac{\mathsf{Pr}(V_h|S_h)\mathsf{Pr}(H_t|V_h)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_h)\mathsf{Pr}(H_t|V)} &> \frac{\mathsf{Pr}(V_h|S_l)\mathsf{Pr}(H_t|V_h)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_l)\mathsf{Pr}(H_t|V)} \frac{\mathsf{Pr}(V_l|S_h)\mathsf{Pr}(H_t|V_l)}{\sum_{\mathbb{V}} \mathsf{Pr}(V|S_h)\mathsf{Pr}(H_t|V)} \\
\Leftrightarrow \mathsf{Pr}(V_l|S_l)\mathsf{Pr}(V_h|S_h) &> \mathsf{Pr}(V_h|S_l)\mathsf{Pr}(V_l|S_h) \\
\Leftrightarrow \mathsf{Pr}(S_l|V_l)\mathsf{Pr}(S_h|V_h) &> \mathsf{Pr}(S_l|V_h)\mathsf{Pr}(S_h|V_l)
\end{aligned}$$

The MLRP is thus dynamically maintained, implying the order of expectation.

(b) First we show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_3)$. Suppose otherwise; thus $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$. Then the MLRP conditions $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_2|V_3) > \mathsf{Pr}(S_2|V_1)\mathsf{Pr}(S_1|V_3)$ and $\mathsf{Pr}(S_1|V_1)\mathsf{Pr}(S_3|V_3) > \mathsf{Pr}(S_3|V_1)\mathsf{Pr}(S_1|V_3)$ imply respectively the following two conditions

$$\begin{aligned}
\mathsf{Pr}(S_2|V_1) &< \mathsf{Pr}(S_2|V_3); \qquad (12) \\
\mathsf{Pr}(S_3|V_1) &< \mathsf{Pr}(S_3|V_3).
\end{aligned}$$

These two conditions, together with $\mathsf{Pr}(S_1|V_1) \leq \mathsf{Pr}(S_1|V_3)$, imply that $\sum_{i=1}^{3} \mathsf{Pr}(S_i|V_3) > \sum_{i=1}^{3} \mathsf{Pr}(S_i|V_1)$. But this contradicts $\sum_{i=1}^{3} \mathsf{Pr}(S_i|V_j) = 1$ for every j.

The same argument can be applied to show that $\mathsf{Pr}(S_1|V_1) > \mathsf{Pr}(S_1|V_2)$ and $\mathsf{Pr}(S_1|V_2) > \mathsf{Pr}(S_1|V_3)$, and also in the reverse direction for $\mathsf{Pr}(S_3|V_1) < \mathsf{Pr}(S_3|V_2) < \mathsf{Pr}(S_3|V_3)$.

Proof of Proposition 2

1. Suppose contrary to the claim, an informed trader with signal S_1 buys at some history H_t. Then by Proposition 1 (a) every informed trader buys at H_t. But this implies that

at history H_t

$$\mathsf{p}_t^A = \mathsf{p}_t^B = \mathsf{E}[V|H_t]$$

But since, by Proposition 1 (a), $\mathsf{E}[V|H_t] > \mathsf{E}[V|S_1, H_t]$ we have that an informed trader with signal S_1 always sells. This is a contradiction.

The proof of the claim that informed traders with signal S_3 always buy is analogous.

(b) We shall prove the result in the case of buys. The same reasoning applies to sales.

Consider any arbitrary history H_t. Then by the previous step there are two possibilities at H_t.

Case 1: Only S_3 types buy. Then for any $V' > V$

$$\mathsf{Pr}(\mathsf{buy}|H_t, V') - \mathsf{Pr}(\mathsf{buy}|H_t, V) = \mu\left(\mathsf{Pr}(S_3|V') - \mathsf{Pr}(S_3|V)\right) > 0.$$

The inequality in the above follows from Proposition 1 (b).

Case 2: Only S_2 and S_3 types buy. Then

$$\begin{aligned}\mathsf{Pr}(\mathsf{buy}|H_t, V') - \mathsf{Pr}(\mathsf{buy}|H_t, V) &= \mu\left(\mathsf{Pr}(S_3|V') + \mathsf{Pr}(S_2|V') - \mathsf{Pr}(S_3|V) - \mathsf{Pr}(S_2|V)\right) \\ &= \mu\left(1 - \mathsf{Pr}(S_1|V') - (1 - \mathsf{Pr}(S_1|V))\right) \\ &= \mu\left(\mathsf{Pr}(S_1|V) - \mathsf{Pr}(S_1|V')\right) > 0,\end{aligned}$$

Again, the inequality in the above follows from Proposition 1 (2). The result follows by setting ϵ to equal

$$\min\left\{\mu\left(\mathsf{Pr}(S_3|V') - \mathsf{Pr}(S_3|V)\right), \mu\left(\mathsf{Pr}(S_1|V) - \mathsf{Pr}(S_1|V')\right)\right\}.$$

Proof of Proposition 4

Denote the public belief at t that the true liquidation value is V_i by

$$q_i^t \equiv \mathsf{Pr}(V_i|H_t)$$

Also denote respectively the probability of a buy and the probability of a sale at t by

$$\beta_i^t \equiv \mathsf{Pr}(\mathsf{buy}|V_i, H_t) \text{ and } \sigma_i^t \equiv \mathsf{Pr}(\mathsf{sale}|V_i, H_t).$$

For the ease of exposition, when the meaning is clear we shall at times denote q_i^t, β_i^t and σ_i^t by q_i, β_i and σ_i.

We also use $\propto$ to denote that two expressions have the same sign; thus for any real numbers x and y the expression $x \propto y$ stands for x and y having the same sign.

Step 1: *For any H_t, we have*

$$\mathsf{E}[V|S_i, H_t] - \mathsf{p}_t^A \quad \propto \quad \begin{cases} [\beta_2^t \mathsf{Pr}(S_i|V_3) - \beta_3^t \mathsf{Pr}(S_i|V_2)] \\ +\frac{q_1^t}{q_3^t}[\beta_1^t \mathsf{Pr}(S_i|V_2) - \beta_2^t \mathsf{Pr}(S_i|V_1)] \\ +\frac{2q_1^t}{q_2^t}[\beta_1^t \mathsf{Pr}(S_i|V_3) - \beta_3^t \mathsf{Pr}(S_i|V_1)] \end{cases} \tag{13}$$

Note that

$$\mathsf{E}[V|S_i, H_t] - \mathsf{p}_t^A = \mathcal{V} q_2 \left(\frac{\mathsf{Pr}(S_i|V_2)}{\mathsf{Pr}(S_i)} - \frac{\beta_2}{\mathsf{Pr}(\mathsf{buy}|H_t)} \right) + 2\mathcal{V} q_3 \left(\frac{\mathsf{Pr}(S_i|V_3)}{\mathsf{Pr}(S_i)} - \frac{\beta_3}{\mathsf{Pr}(\mathsf{buy}|H_t)} \right)$$

But the RHS of the above has the same sign as

$$\begin{aligned} & \mathsf{Pr}(S_i|V_2) \sum_j \beta_j q_j - \beta_2 \sum_j \mathsf{Pr}(S_i|V_j) q_j + 2 \, \frac{q_3}{q_2} \left(\mathsf{Pr}(S_i|V_3) \sum_j \beta_j q_j - \beta_3 \sum_j \mathsf{Pr}(S_i|V_j) q_j \right) \\ = \; & q_1 \left(\beta_1 \mathsf{Pr}(S_i|V_2) - \beta_2 \mathsf{Pr}(S_i|V_1) \right) + q_3 \left(\beta_3 \mathsf{Pr}(S_i|V_2) - \beta_2 \mathsf{Pr}(S_i|V_3) \right) \\ & +2 \, \frac{q_3}{q_2} \left(q_1 \; \left(\beta_1 \mathsf{Pr}(S_i|V_3) - \beta_3 \mathsf{Pr}(S_i|V_1) \right) + q_2 \left(\beta_2 \mathsf{Pr}(S_i|V_3) - \beta_3 \mathsf{Pr}(S_i|V_2) \right) \right) \end{aligned}$$

But this implies that

$$\mathsf{E}[V|S_i] - \mathsf{p}^A \propto \begin{cases} [\beta_2 \mathsf{Pr}(S_i|V_3) - \beta_3 \mathsf{Pr}(S_i|V_2)] \\ +\frac{q_1}{q_3}[\beta_1 \mathsf{Pr}(S_i|V_2) - \beta_2 \mathsf{Pr}(S_i|V_1)] \\ +2 \; \frac{q_1}{q_2}[\beta_1 \mathsf{Pr}(S_i|V_3) - \beta_3 \mathsf{Pr}(S_i|V_1)]. \end{cases}$$

Step 2: *For any H_t, we have*

$$\mathsf{E}[V|S_i, H_t] - \mathsf{p}_t^B \quad \propto \quad \begin{cases} [\sigma_2^t \mathsf{Pr}(S_i|V_3) - \sigma_3^t \mathsf{Pr}(S_i|V_2)] \\ +\frac{q_1^t}{q_3^t}[\sigma_1^t \mathsf{Pr}(S_i|V_2) - \sigma_2^t \mathsf{Pr}(S_i|V_1)] \\ +2 \; \frac{q_1^t}{q_2^t}[\sigma_1^t \mathsf{Pr}(S_i|V_3) - \sigma_3^t \mathsf{Pr}(S_i|V_1)]. \end{cases}$$

This follows by analogous arguments as in Step 1.

Step 3: $\mathsf{E}[V|S_2] - \mathsf{p}_1^B < 0$ *if and only if* $\mu < \mu_b^{in}$.
Then since $\mathsf{Pr}(V_1) = \mathsf{Pr}(V_3)$ it follows from Step 2 that $\mathsf{E}[V|S_i] - \mathsf{p}_1^B < 0$ is equivalent to

$$\Leftrightarrow \quad \begin{array}{r} [\sigma_2^1 \mathsf{Pr}(S_i|V_3) - \sigma_3^1 \mathsf{Pr}(S_i|V_2)] + [\sigma_1^1 \mathsf{Pr}(S_i|V_2) - \sigma_2^1 \mathsf{Pr}(S_i|V_1)] \\ +2 \, \frac{\mathsf{Pr}(V_1)}{\mathsf{Pr}(V_2)} [\sigma_1^1 \mathsf{Pr}(S_i|V_3) - \sigma_3^1 \mathsf{Pr}(S_i|V_1)] < 0 \end{array} \tag{14}$$

Since S_3-types always buy we have that

$$\sigma_i^1 = \begin{cases} \gamma + \mu \mathsf{Pr}(S_1|V_i) & \text{if } S_2\text{-types does not sell at } \mathsf{p}_1^B \\ \gamma + \mu(\mathsf{Pr}(S_1|V_i) + \mathsf{Pr}(S_2|V_i)) & \text{if } S_2\text{-types sells at } \mathsf{p}_t^B. \end{cases}$$

Substituting for $\sigma_i^1, i = 1, 2, 3$ in (14) and simplifying we have

$$\mathsf{E}[V|S_2] - \mathsf{p}_1^B < 0 \ \Leftrightarrow \ \frac{\mu}{\gamma} < \theta_b.$$

(the last expression holds irrespective of whether S_2-types sells or not at p_t^B). But this implies that $\mathsf{E}[V|S_i] - \mathsf{p}_1^B < 0$ if and only if $\mu < \frac{\theta_b}{3+\theta_b} = \mu_b^{in}$.

Step 4: *For any* $\eta > 0$ *there exists a history* H_t *consisting of only buys such that* $\frac{q_1^t}{q_2^t} < \eta$ *and* $\frac{q_1^t}{q_3^t} < \eta$.
Since by Proposition 2 (b) there exists $\epsilon > 0$ such that $\mathsf{Pr}(\mathsf{buy}|V_j, H_t) > \mathsf{Pr}(\mathsf{buy}|V_i, H_t) + \epsilon$ for any history H_t and any $i, j = 1, 2, 3$ and $j > i$, it follows that for sufficiently large t any history H_t consisting only of buys is such that $\frac{q_1^t}{q_2^t} < \eta$ and $\frac{q_1^t}{q_3^t} < \eta$.

Step 5: For any date t we have $\beta_2^t \mathsf{Pr}(S_i|V_3) - \beta_3^t \mathsf{Pr}(S_i|V_2) = \rho_{23}^{23}\left(\frac{k_b(1-\mu)}{3} - \mu\right)$.
Since S_1-types always sell we have that

$$\beta_i^t = \begin{cases} \gamma + \mu \mathsf{Pr}(S_3|V_i) & \text{if } S_2\text{-types does not buyat } \mathsf{p}_1^A \\ \gamma + \mu\left(\mathsf{Pr}(S_3|V_i) + \mathsf{Pr}(S_2|V_i)\right) & \text{if } S_2\text{-types buys at } \mathsf{p}_1^A. \end{cases}$$

This implies, irrespective of whether S_2-types buy or not at p_1^A, that

$$\begin{aligned} & \{\beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2)\} \\ = \; & \gamma\left(\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)\right) + \mu\left(\mathsf{Pr}(S_3|V_2)\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_3|V_3)\mathsf{Pr}(S_2|V_2)\right) \\ = \; & \frac{(1-\mu)\left(\mathsf{Pr}(S_2|V_3) - \mathsf{Pr}(S_2|V_2)\right)}{3} - \mu\rho_{23}^{23} = \rho_{23}^{23}\left(\frac{k_b(1-\mu)}{3} - \mu\right). \end{aligned}$$

Step 6: $\mathsf{E}[V|S_2,H_t]-\mathsf{p}_t^A>0$ *and* $\mathsf{E}[V|H_t]>\mathsf{E}[V]$ *for some history* H_t *if* $\mu<\mu_b^{sw}$.
Suppose that $\mu+\eta<\frac{k_b}{3+k_b}$ for some $\eta>0$. By Step 4, there exist a history H_t of only buys such that the sum of the second and the third term on the RHS of (6) is less than $\eta\rho_{23}^{23}(1+\frac{k_b}{3})$:

$$\frac{q_1^t}{q_3^t}[\beta_1^t\mathsf{Pr}(S_i|V_2)-\beta_2^t\mathsf{Pr}(S_i|V_1)]+\frac{2q_1^t}{q_2^t}[\beta_1^t\mathsf{Pr}(S_i|V_3)-\beta_3^t\mathsf{Pr}(S_i|V_1)]<\eta\rho_{23}^{23}(1+\frac{k_b}{3}). \quad (15)$$

Since $\mu+\eta<\frac{k_b}{3+k_b}$ it follows that $\eta(1+\frac{k_b}{3})<\frac{k_b(1-\mu)}{3}-\mu$. But then, by from Step 5, the first term on the RHS of (6), $[\beta_2^t\mathsf{Pr}(S_i|V_3)-\beta_3^t\mathsf{Pr}(S_i|V_2)]$, exceeds $\eta\rho_{23}^{23}(1+\frac{k_b}{3})$. But this, together with (15), establish that $\mathsf{E}[V|S_2,H_t]-\mathsf{p}_t^A>0$.

To show that $\mathsf{E}[V|H_t]>\mathsf{E}[V]$, note that

$$\begin{aligned}\mathsf{E}[V|H_t]-\mathsf{E}[V] &= \mathcal{V}\{(1-q_1^t-q_3^t)+2q_3^t\}-\mathcal{V} \\ &= \mathcal{V}(q_3^t-q_1^t)\end{aligned} \quad (16)$$

This, together with $\frac{q_1^t}{q_3^t}<\eta<1$, imply that $\mathsf{E}[V|H_t]>\mathsf{E}[V]$.

Step 7: *If* $\mathsf{E}[V|S_2]-\mathsf{p}_1^B<0$, $\mathsf{E}[V|S_2,H_t]-\mathsf{p}_t^A>0$ *and* $\mathsf{E}[V|H_t]>\mathsf{E}[V]$ *for some history* H_t *then* $\mu<\frac{k_b}{3+k_b}$.
By Step 1 and $\mathsf{E}[V|S_2,H_t]-\mathsf{p}_t^A>0$ we have

$$\begin{array}{l}[\beta_2\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_2)]+\frac{q_1}{q_3}[\beta_1\mathsf{Pr}(S_2|V_2)-\beta_2\mathsf{Pr}(S_2|V_1)] \\ +\frac{2q_1}{q_2}[\beta_1\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_1)].\end{array} >0 \quad (17)$$

Now since $\mathsf{E}[V|S_2]-\mathsf{p}_1^B<0$ by Step 3 we have $0<\mu<\frac{\theta_b}{3+\theta_b}$. Therefore, $\theta_b>0$ and hence

$$\mathsf{Pr}(S_2|V_3)-\mathsf{Pr}(S_2|V_1)<0 \quad (18)$$

But then, by condition (17), we have

$$\beta_2\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_2)+\frac{q_1^t}{q_3^t}[\beta_1\mathsf{Pr}(S_2|V_2)-\beta_2\mathsf{Pr}(S_2|V_1)]>0 \quad (19)$$

This together with (18) and $0<\beta_1<\beta_2<\beta_3$ (Proposition 2 (b)) imply that

$$\begin{aligned}\beta_2\mathsf{Pr}(S_2|V_1)-\beta_1\mathsf{Pr}(S_2|V_2)+\frac{q_1^t}{q_3^t}[\beta_1\mathsf{Pr}(S_2|V_2)-\beta_2\mathsf{Pr}(S_2|V_1)] &> 0 \\ \Rightarrow(\beta_2\mathsf{Pr}(S_2|V_1)-\beta_1\mathsf{Pr}(S_2|V_2))\left(1-\frac{q_1^t}{q_3^t}\right) &> 0\end{aligned} \quad (20)$$

Also, since $\mathsf{E}[V|H_t] > \mathsf{E}[V]$ it follows from (16) that

$$q_3^t - q_1^t > 0$$

This together with (20) imply that

$$\beta_2 \mathsf{Pr}(S_2|V_1) - \beta_1 \mathsf{Pr}(S_2|V_2) > 0.$$

Hence, by (19), we have

$$\beta_2 \mathsf{Pr}(S_2|V_3) - \beta_3 \mathsf{Pr}(S_2|V_2) > 0. \tag{21}$$

But then by Step 5 we have $k_b(1-\mu) - 3\mu > 0$; hence $\mu < \frac{k_b}{3+k_b}$. This completes the proof of this step.

Claims (a) and (b) in the Proposition follow immediately from Steps 3, 6 and 7.

Proof of Proposition 5

We shall prove the result for the case of buy-herding (S_2 being negatively biased) with market expectation approaching V_3; the proof for the other case is analogous.

By Proposition 4 with a positive probability there exists a history H^τ at which buy-herding occurs. By Step 1 in the proof of Proposition 4

$$\begin{array}{l} \beta_2^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_2) + \frac{q_1^t}{q_3^t}\left[\beta_1^t \mathsf{Pr}(S_2|V_2) - \beta_2^t \mathsf{Pr}(S_2|V_1)\right] \\ + \frac{2q_1^t}{q_2^t}\left[\beta_1^t \mathsf{Pr}(S_2|V_3) - \beta_3^t \mathsf{Pr}(S_2|V_1)\right] \end{array} > 0. \tag{22}$$

for $t = \tau$. Moreover, buy-herding persists if (22) holds for any $t > \tau$. Since buy-herding occurs at H^τ we know that the first term in the above expression is positive while the second and the third terms are negative. This implies that after date τ (22) will continue to hold, as long as $\frac{q_1^t}{q_3^t}$ and $\frac{q_1^t}{q_2^t}$ are non-increasing. This is indeed the case if the unfolding history involves only buys because, by Proposition 2 (b), $\beta_j^t > \beta_i^t$ at any history and for any $i, j = 1, 2, 3$ and $j > i$. This directly implies that buys decrease coefficients $\frac{q_1^t}{q_3^t}$ and $\frac{q_1^t}{q_2^t}$, and in the limit we have

$$\frac{q_1^t}{q_3^t} \to 0 \text{ and } \frac{q_1^t}{q_2^t} \to 0.$$

Thus for continuing buys, herding persists beyond period τ.

We now show that for continuing buys, beyond period τ, the prices (during this buy-herding phase) will approach V_3 . To see this first note that the S_2-insider's expectation is above the market maker's. Thus if the upper bound for the public expectation during herding is V_3, so it is for the S_2-insider's expectation. To see observe that

$$\mathsf{E}[V|H_t] = \sum_i V_i q_i = \frac{1}{q_3^t}\left(V_2 \frac{q_2^t}{q_3^t} + V_3\right).$$

But since by Proposition 2 (b), there exists $\epsilon > 0$ such that $\beta_3^t > \beta_2^t + \epsilon$ for every history H^t (irrespective of whether there is herding or not) it follows that $\frac{q_2^t}{q_3^t} \to 0$ for any history H^t involving only buys after H^τ as t goes to infinity. Consequently, for every $\epsilon > 0$, exists a finite (but "long") history consisting of H^τ followed by sufficiently many (herd-)buys such that

$$\mathsf{E}[V|H_t] > V_3 - \epsilon.$$

To prove (c), observe that as $q_2^0 \to 1$, μ_b may change (if $\mu_b = \mu_b^{in}$ it remains constant), but it is always bounded away from zero ($\theta_b \neq 0$). So let μ_* be the limit of μ_b as $q_2^0 \to 1$ and consider any $\mu < \mu_*$. As $q_2^0 \to 1$, (6) goes to

$$[\beta_2^t \mathsf{Pr}(S_i|V_3) - \beta_3^t \mathsf{Pr}(S_i|V_2)] + \frac{q_1^t}{q_3^t}[\beta_1^t \mathsf{Pr}(S_i|V_2) - \beta_2^t \mathsf{Pr}(S_i|V_1)].$$

Let λ^* solve

$$[\beta_2^t \mathsf{Pr}(S_i|V_3) - \beta_3^t \mathsf{Pr}(S_i|V_2)] + \lambda^*[\beta_1^t \mathsf{Pr}(S_i|V_2) - \beta_2^t \mathsf{Pr}(S_i|V_1)] = 0.$$

Then in the limit case, buy-herding starts once $\frac{q_1^t}{q_3^t} < \lambda^*$. Consider a history that consists only of b no-herd buys. Then

$$\frac{q_1^t}{q_3^t} = \left(\frac{\mu \mathsf{Pr}(S_3|V_1) + \gamma}{\mu \mathsf{Pr}(S_3|V_3) + \gamma}\right)^b \quad \Leftrightarrow \quad \exists b^* \in \mathbb{R} \text{ such that } \left(\frac{\mu \mathsf{Pr}(S_3|V_1) + \gamma}{\mu \mathsf{Pr}(S_3|V_3) + \gamma}\right)^{b^*} = \lambda^*.$$

Define $\lceil b^* \rceil$ to be the smallest integer larger than b^*, so that after $\lceil b^* \rceil$ buys, herding starts. When $q_2^0 < 1$, the third term in (6) is non-zero, but as $q_2^0 \to 1$, the term gets arbitrarily small. Thus there exists a threshold $\bar{q}_2^0$ so that for all $q_2^0 > \bar{q}_2^0$, after exactly $\lceil b^* \rceil$ no-herd buys, (6) is indeed positive. In other words, as the probabilities of V_1 and V_3 are sufficiently small, the number of trades necessary for herding is $\lceil b^* \rceil$. Finally, using that $q_2^0 = 1 - \frac{1}{n}$, the public expectation after $\bar{b}$ buys

$$\mathsf{E}[V|H_t = \{\bar{b} \text{ no-herd buys}\}] \to_{n\to\infty} V_2.$$

Then for any ϵ, exists n so that $\mathsf{E}[V|H_t = \{\bar{b} \text{ no-herd buys}\}] \in (V_2, V_2 + \epsilon)$.

Proof of Proposition 6

The fourth part of Proposition 6, (d), is trivially true. To prove $(a)-(c)$, we first show that for any $H_t = (a_1, \ldots, a_{r+b+s})$ satisfying the above we have that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t]$ has the same sign as

$$[(\beta_3\beta_{2,n})^b(\sigma_3\sigma_{2,n})^s - (\beta_{3,n}\beta_2)^b(\sigma_{3,n}\sigma_2)^s] + \frac{q_1^r}{q_3^r}[(\beta_2\beta_{1,n})^b(\sigma_2\sigma_{1,n})^s - (\beta_{2,n}\beta_1)^b(\sigma_{2,n}\sigma_1)^s] \quad (23)$$
$$+2\,\frac{q_1^r}{q_2^r}[(\beta_3\beta_{1,n})^b(\sigma_3\sigma_{1,n})^s - (\beta_{3,n}\beta_1)^b(\sigma_{3,n}\sigma_1)^s].$$

To see that (23) holds, observe

$$\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] = \mathcal{V}\{(q_2^t - q_{2,n}^t) + 2(q_3^t - q_{3,n}^t)\}$$
$$= \mathcal{V}\left\{q_2^r\left(\frac{\beta_2^b\sigma_2^s}{\sum_i q_i^r\beta_i^b\sigma_i^s} - \frac{\beta_{2,n}^b\sigma_{2,n}^s}{\sum_i q_i^r\beta_{i,n}^b\sigma_{i,n}^s}\right) + 2q_3^r\left(\frac{\beta_3^b\sigma_3^s}{\sum_i q_i^r\beta_i^b\sigma_i^s} - \frac{\beta_{3,n}^b\sigma_{3,n}^s}{\sum_i q_i^r\beta_{i,n}^b\sigma_{i,n}^s}\right)\right\}$$

Therefore,

$$\begin{aligned}\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] \;\propto\;& q_2^r q_1^r[(\beta_2\beta_{1,n})^b(\sigma_2\sigma_{1,n})^s - (\beta_{2,n}\beta_1)^b(\sigma_{2,n}\sigma_1)^s]\\ &+2q_3^r q_1^r[(\beta_3\beta_{1,n})^b(\sigma_3\sigma_{1,n})^s - (\beta_{3,n}\beta_1)^b(\sigma_{3,n}\sigma_1)^s]\\ &+q_3^r q_2^r[(\beta_3\beta_{2,n})^b(\sigma_3\sigma_{2,n})^s - (\beta_{3,n}\beta_2)^b(\sigma_{3,n}\sigma_2)^s]\end{aligned}$$

Dividing the RHS of the above and rearranging we have

$$\begin{aligned}\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] \;\propto\;& [(\beta_3\beta_{2,n})^b(\sigma_3\sigma_{2n})^s - (\beta_{3,n}\beta_2)^b(\sigma_{3,n}\sigma_2)^s] \quad (24)\\ &+\frac{q_1^r}{q_3^r}[(\beta_2\beta_{1,n})^b(\sigma_2\sigma_{1,n})^s - (\beta_{2,n}\beta_1)^b(\sigma_{2,n}\sigma_1)^s]\\ &+\frac{2q_1^r}{q_2^r}[(\beta_3\beta_{1,n})^b(\sigma_3\sigma_{1,n})^s - (\beta_{3,n}\beta_1)^b(\sigma_{3,n}\sigma_1)^s]\end{aligned}$$

We can now manipulate (23) and write that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t]$ has the same sign as

$$\begin{aligned}&[(\beta_3\beta_{2,n})^b - (\beta_{3,n}\beta_2)^b](\sigma_3\sigma_{2,n})^s + [(\sigma_3\sigma_{2,n})^s - (\sigma_{3,n}\sigma_2)^s](\beta_{3,n}\beta_2)^b\\ &+\frac{q_1^r}{q_3^r}\left\{[(\beta_2\beta_{1,n})^b - (\beta_{2,n}\beta_1)^b](\sigma_2\sigma_{1,n})^s + [(\sigma_2\sigma_{1,n})^s - (\sigma_{2,n}\sigma_1)^s](\beta_{2,n}\beta_1)^b\right\}\\ &+\frac{2q_1^r}{q_2^r}\left\{[(\beta_3\beta_{1,n})^b - (\beta_{3,n}\beta_1)^b](\sigma_3\sigma_{1,n})^s + [(\sigma_3\sigma_{1,n})^s - (\sigma_{3,n}\sigma_1)^s](\beta_{3,n}\beta_1)^b\right\}\end{aligned}$$

Next we expand the above to obtain that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t]$ has the same sign as

$$\begin{array}{c}\left\{\begin{array}{c}(\sigma_3\sigma_{2,n})^s[\beta_3\beta_{2,n}-\beta_{3,n}\beta_2]\sum_{\tau=0}^{b-1}(\beta_3\beta_{2,n})^{b-1-\tau}(\beta_{3,n}\beta_2)^\tau \\ +(\beta_{3,n}\beta_2)^b[\sigma_3\sigma_{2,n}-\sigma_{3,n}\sigma_2]\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^\tau\end{array}\right\} \\ +\frac{q_1^r}{q_3^r}\left\{\begin{array}{c}(\sigma_2\sigma_{1,n})^s[\beta_2\beta_{1,n}-\beta_{2,n}\beta_1]\sum_{\tau=0}^{b-1}(\beta_2\beta_{1,n})^{b-1-\tau}(\beta_{2,n}\beta_1)^\tau \\ +(\beta_{2,n}\beta_1)^b[\sigma_2\sigma_{1,n}-\sigma_{2,n}\sigma_1]\sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,n})^{s-1-\tau}(\sigma_{2,n}\sigma_1)^\tau\end{array}\right\} \\ +2\ \frac{q_1^r}{q_2^r}\left\{\begin{array}{c}(\sigma_3\sigma_{1,n})^s[\beta_3\beta_{1,n}-\beta_{3,n}\beta_1]\sum_{\tau=0}^{b-1}(\beta_3\beta_{1,n})^{b-1-\tau}(\beta_3\beta_1)^\tau \\ +(\beta_{3,n}\beta_1)^b[\sigma_3\sigma_{1,n}-\sigma_{3,n}\sigma_1]\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,n})^{s-1-\tau}(\sigma_{3,n}\sigma_1)^\tau\end{array}\right\}\end{array} \tag{25}$$

To prove (a), we show that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] > 0$ *if* $s = 0$.
Since $\beta_3 > \beta_2 > \beta_1$ and $\beta_{3,n} > \beta_{2,n} > \beta_{1,n}$ it follows that

$$\begin{aligned}\sum_{\tau=0}^{b-1}(\beta_3\beta_{2,n})^{b-1-\tau}(\beta_{3,n}\beta_2)^\tau &> \sum_{\tau=0}^{b-1}(\beta_2\beta_{1,n})^{b-1-\tau}(\beta_{2,n}\beta_1)^\tau \\ \sum_{\tau=0}^{b-1}(\beta_3\beta_{2,n})^{b-1-\tau}(\beta_{3,n}\beta_2)^\tau &> \sum_{\tau=0}^{b-1}(\beta_3\beta_{1,n})^{b-1-\tau}(\beta_3\beta_1)^\tau\end{aligned}$$

Since there is herding at r, by Step 1 of the proof of Proposition 4 we have

$$\begin{array}{r}[\beta_2\mathsf{Pr}(S_i|V_3)-\beta_3\mathsf{Pr}(S_i|V_2)]+\frac{q_1^r}{q_3^t}[\beta_1\mathsf{Pr}(S_i|V_2)-\beta_2\mathsf{Pr}(S_i|V_1)] \\ +2\ \frac{q_1^r}{q_2^r}[\beta_1\mathsf{Pr}(S_i|V_3)-\beta_3\mathsf{Pr}(S_i|V_1)]\end{array} > 0 \tag{26}$$

Also, note that

$$\begin{aligned}\beta_3\beta_{2,n}-\beta_{3,n}\beta_2 &= \mu[\beta_2\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_2)] > 0 \\ \beta_3\beta_{1,n}-\beta_{3,n}\beta_1 &= \mu[\beta_1\mathsf{Pr}(S_2|V_3)-\beta_3\mathsf{Pr}(S_2|V_1)] < 0 \\ \beta_2\beta_{1,n}-\beta_{2,n}\beta_1 &= \mu[\beta_1\mathsf{Pr}(S_2|V_2)-\beta_2\mathsf{Pr}(S_2|V_1)] < 0\end{aligned} \tag{27}$$

(The inequalities follow from $0 < \mu < \mu_b$.) Therefore, using (26) we have

$$[\beta_3\beta_{2,n}-\beta_{3,n}\beta_2]+\frac{q_1^r}{q_3^t}[\beta_2\beta_{1,n}-\beta_{2,n}\beta_1]+\frac{2q_1^r}{q_2^r}[\beta_3\beta_{1,n}-\beta_{3,n}\beta_1] > 0 \tag{28}$$

By (27) and (6), this implies that for $s = 0$, (25) is positive, because (25) is just

$$
\begin{aligned}
&[(\beta_3\beta_{2,n}) - (\beta_{3,n}\beta_2)] \sum_{\tau=0}^{b-1} (\beta_3\beta_{2,n})^{b-1-\tau}(\beta_{3,n}\beta_2)^{\tau} \\
&+\frac{q_1^r}{q_3^r}\left\{(\beta_2\beta_{1,n} - \beta_{2,n}\beta_1)\sum_{\tau=0}^{b-1}(\beta_2\beta_{1,n})^{b-1-\tau}(\beta_{2,n}\beta_1)^{\tau}\right\} \\
&+2\frac{q_1^r}{q_2^r}\left\{(\beta_3\beta_{1,n} - \beta_{3,n}\beta_1)\sum_{\tau=0}^{b-1}(\beta_3\beta_{1,n})^{b-1-\tau}(\beta_3\beta_1)^{\tau}\right\} > 0
\end{aligned}
$$

But this implies that $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] > 0$ for $s = 0$.

We now show (b), i.e. that if $b = 0$ *and* $\mathsf{E}[V|S_2, H_t] > \mathsf{p}_t^B$ *then* $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t] < 0$. First, since $\sigma_1 > \sigma_2 > \sigma_3$ and $\sigma_{1,n} > \sigma_{2,n} > \sigma_{3,n}$ it follows that

$$
\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^{\tau} < \sum_{\tau=0}^{s-1}(\sigma_2\sigma_{1,n})^{s-1-\tau}(\sigma_{2,n}\sigma_1)^{\tau} \tag{29}
$$

$$
\sum_{\tau=0}^{s-1}(\sigma_3\sigma_{2,n})^{s-1-\tau}(\sigma_{3,n}\sigma_2)^{\tau} < \sum_{\tau=0}^{s-1}(\sigma_3\sigma_{1,n})^{s-1-\tau}(\sigma_{3,n}\sigma_1)^{\tau} \tag{30}
$$

Next, since there is herding at r it must be that $\mathsf{E}[V|S_2, H_r] > \mathsf{p}_r^B$. By Step 2 in the proof of Proposition 4 this implies

$$
\begin{aligned}
&[\sigma_2 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_2)] + \tfrac{q_1^r}{q_3^r}[\sigma_1 \mathsf{Pr}(S_2|V_2) - \sigma_2 \mathsf{Pr}(S_2|V_1)] \\
&\qquad + 2\,\tfrac{q_1^r}{q_2^r}[\sigma_1 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_1)] > 0
\end{aligned} \tag{31}
$$

Simple computation shows that

$$
\begin{aligned}
\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2 &= -\mu[\sigma_2 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_2)] \\
&= -\mu^2\rho_{12}^{23} + \mu\gamma(\mathsf{Pr}(S_2|V_2) - \mathsf{Pr}(S_2|V_3)) < 0 \\
\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1 &= -\mu[\sigma_1 \mathsf{Pr}(S_2|V_3) - \sigma_3 \mathsf{Pr}(S_2|V_1)] \\
&< -\mu[\sigma_1 \mathsf{Pr}(S_2|V_2) - \sigma_2 \mathsf{Pr}(S_2|V_1)] \\
&= \sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1
\end{aligned} \tag{32}
$$

Therefore, (31) is equivalent to

$$
[\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2] + \frac{q_1^r}{q_3^r}[\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] + \frac{2q_1^r}{q_2^r}[\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1] < 0, \tag{33}
$$

where the LHS is merely (24) with $b = 0$. The first term on LHS is negative, whereas at first blush, the signs of the second and third term seem indeterminate.

However, at time 0, we require (31) to be negative as the S_2-type's expectation must be below the bid-price. Then (33) must be positive, and thus at least one term on the LHS must be positive. Moreover, by (32) at least the second term must be positive, $\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1 > 0$. So far the sign of the third term is indeterminate, but requiring $\mu > \mu_{hb}$, ensures it is negative. In summary,

$$[\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] > 0 > [\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1] \tag{34}$$

Consider now the situation after s sales. If $s = 1$, the claim is obviously true since (33) and (31) have the same sign. Suppose now, that $s > 1$ and that herding prevails. Then (31) can be written as

$$[\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2] + \frac{q_1^r}{q_3^r}\left(\frac{\sigma_1}{\sigma_3}\right)^s [\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] + \frac{2q_1^r}{q_2^r}\left(\frac{\sigma_1}{\sigma_2}\right)^s [\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1] \tag{35}$$

Likewise, since $b = 0$, we can write (25) as

$$\begin{array}{c}[(\sigma_3\sigma_{2,n}) - (\sigma_{3,n}\sigma_2)] \sum_{\tau=0}^{s}(\sigma_3\sigma_{2,n})^{s-\tau}(\sigma_{3,n}\sigma_2)^{\tau} \\ +\frac{q_1^r}{q_3^r}\left\{(\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1)\sum_{\tau=0}^{s}(\sigma_2\sigma_{1,n})^{s-\tau}(\sigma_{2,n}\sigma_1)^{\tau}\right\} \\ +2\frac{q_1^r}{q_2^r}\left\{(\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1)\sum_{\tau=0}^{s}(\sigma_3\sigma_{1,n})^{s-\tau}(\sigma_3\sigma_1)^{\tau}\right\}.\end{array}$$

This expression can be rearranged to be

$$\begin{array}{r}[\sigma_3\sigma_{2,n} - \sigma_{3,n}\sigma_2] + \frac{q_1^r}{q_3^r}\frac{\sum_{\tau=0}^{s}(\sigma_2\sigma_{1,n})^{s-\tau}(\sigma_{2,n}\sigma_1)^{\tau}}{\sum_{\tau=0}^{s}(\sigma_3\sigma_{2,n})^{s-\tau}\;(\sigma_{3,n}\sigma_2)^{\tau}}[\sigma_2\sigma_{1,n} - \sigma_{2,n}\sigma_1] \\ +\frac{2q_1^r}{q_2^r}\frac{\sum_{\tau=0}^{s}(\sigma_3\sigma_{1,n})^{s-\tau}(\sigma_3\sigma_1)^{\tau}}{\sum_{\tau=0}^{s}(\sigma_3\sigma_{2,n})^{s-\tau}(\sigma_{3,n}\sigma_2)^{\tau}}[\sigma_3\sigma_{1,n} - \sigma_{3,n}\sigma_1]\end{array} \tag{36}$$

Since (35) is a rewriting of $\mathsf{p}_t^B - \mathsf{E}[V|S_2, H_{r+b+s}]$, and (36) is a rewriting of $\mathsf{E}[V|H_t] - \mathsf{E}_n[V|H_t]$. Our aim is to learn the sign of (36) from the sign of (35), which we know to be negative by assumption.

The first terms in both (35) and (36) are identical, the second terms are positive, the third terms are negative. Sales will increase the weights on both the second term, and third terms. Of course, the effect on the second terms will always be stronger as otherwise the herd would never be broken. Thus sales put "strain" on the inequality being negative.

We now argue that the effect of the sales on the negative second term in (35) is stronger than in (36) and the effect on the positive third term is smaller in (35) than in (36). Both together ensure that if the LHS in (35) is negative, so is the LHS in (36). Formally,

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s}(\sigma_2\sigma_{1,n})^{s-\tau}(\sigma_{2,n}\sigma_1)^\tau}{\sum_{\tau=0}^{s}(\sigma_3\sigma_{2,n})^{s-\tau}(\sigma_{3,n}\sigma_2)^\tau}, \quad \left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s}(\sigma_3\sigma_{1,n})^{s-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s}(\sigma_3\sigma_{2,n})^{s-\tau}(\sigma_{3,n}\sigma_2)^\tau}. \tag{37}$$

Simple manipulations of the first expression show that

$$\left(\frac{\sigma_1}{\sigma_3}\right)^s > \frac{\sum_{\tau=0}^{s}(\sigma_2\sigma_{1,n})^{s-\tau}(\sigma_{2,n}\sigma_1)^\tau}{\sum_{\tau=0}^{s}(\sigma_3\sigma_{2,n})^{s-\tau}(\sigma_{3,n}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s}{\sigma_2}^{s-\tau}{\sigma_{2,n}}^{\tau}(\sigma_1\sigma_3)^\tau\left((\sigma_1\sigma_{3,n})^{s-\tau}-(\sigma_3\sigma_{1,n})^{s-\tau}\right) > 0,$$

and the last inequality is satisfied since $\sigma_1\sigma_{3,n} > \sigma_3\sigma_{1,n}$. Likewise, the second inequality in (37) simplifies to

$$\left(\frac{\sigma_1}{\sigma_2}\right)^s < \frac{\sum_{\tau=0}^{s}(\sigma_3\sigma_{1,n})^{s-\tau}(\sigma_3\sigma_1)^\tau}{\sum_{\tau=0}^{s}(\sigma_3\sigma_{2,n})^{s-\tau}(\sigma_{3,n}\sigma_2)^\tau} \Leftrightarrow \sum_{\tau=0}^{s}{\sigma_3}^{s-\tau}{\sigma_{3,n}}^{\tau}(\sigma_1\sigma_2)^\tau\left((\sigma_2\sigma_{1,n})^{s-\tau}-(\sigma_1\sigma_{2,n})^{s-\tau}\right) > 0,$$

and the last inequality holds because of (34). Consequently, sales have a stronger effect on the difference between bid-price and S_2-expectation than on the difference between the sale transaction prices (the public expectations). For further sales, the result holds recursively: increasing s by single integers, in (36) we can substitute the coefficients before the second and third terms with the more extreme coefficients from (35).

In summary, bid-prices in the rational case stay below prices in the naïve case.

For (c) we reformulate (24) as

$$\begin{aligned}&\frac{(\beta_3\beta_{2,n})^b}{(\beta_{3,n}\beta_1)^b}\left[(\sigma_3\sigma_{2n})^s-\frac{(\beta_{3,n}\beta_2)^b}{(\beta_3\beta_{2,n})^b}(\sigma_{3,n}\sigma_2)^s\right]+\frac{q_1^r}{q_3^r}\frac{(\beta_{2,n}\beta_1)^b}{(\beta_{3,n}\beta_1)^b}\left[\frac{(\beta_2\beta_{1,n})^b}{(\beta_{2,n}\beta_1)^b}(\sigma_2\sigma_{1,n})^s-(\sigma_{2,n}\sigma_1)^s\right]\\&+\frac{2q_1^r}{q_2^r}\left[\frac{(\beta_3\beta_{1,n})^b}{(\beta_{3,n}\beta_1)^b}(\sigma_3\sigma_{1,n})^s-(\sigma_{3,n}\sigma_1)^s\right].\end{aligned} \tag{38}$$

For each term in (24) we factored out the parts that relate to buys. By (27), these were the largest parts. We then divided the entire expression by $(\beta_{3,n}\beta_1)^b$.

Again by (27), for $b \to \infty$, the first term is positive whereas the second and third terms are negative. Since $\beta_{2,n}\beta_1 < \beta_{3,n}\beta_1$, the second term vanishes as $b \to \infty$.

To render the entire equation (38) negative, we need that the first term also vanishes, $\beta_3\beta_{2,n} < \beta_{3,n}\beta_1$: this be the case, all terms but $-(\sigma_{3,n}\sigma_1)^s$ vanish. Simple manipulation of $\beta_3\beta_{2,n} - \beta_{3,n}\beta_1$ shows that the conditions stated in the proposition straightforwardly ensure this difference to be negative.

Proof of Proposition 7

For monotonicity in the sense of Definition 3, we need to understand $\mathsf{E}[V|S_2,H_t]-\mathsf{E}[V|H_t]$. This relation can be derived similarly to equation (6) and it is

$$\mathsf{E}[V|S_i,H_t]-\mathsf{E}[V|H_t] \propto \begin{array}{l} \mathsf{Pr}(S_i|V_3)-\mathsf{Pr}(S_i|V_2)+\frac{q_1^t}{q_3^t}[\mathsf{Pr}(S_i|V_2)-\mathsf{Pr}(S_i|V_1)] \\ +\frac{2q_1^t}{q_2^t}[\mathsf{Pr}(S_i|V_3)-\mathsf{Pr}(S_i|V_1)]. \end{array} \tag{39}$$

If S_2's distribution is monotonic in V, then it is immediately obvious that

$$\mathsf{Pr}(S_2|V_3)-\mathsf{Pr}(S_2|V_2),\ \mathsf{Pr}(S_2|V_2)-\mathsf{Pr}(S_2|V_1),\ \text{and } \mathsf{Pr}(S_2|V_3)-\mathsf{Pr}(S_2|V_1)$$

have the same sign. Consequently, examining the RHS of (39) with $S_i = S_2$, for any trading history, $\mathsf{E}[V|S_2,H_t]-\mathsf{E}[V|H_t]$ will be either always positive (if S_2's distribution is monotonically increasing in V) or always negative (if S_2's distribution is monotonically decreasing in V).

B Distributions Used for Numerical Computations

For Figure 1 we use

$\kappa_b/(3+\kappa_b) = 0.7656$
$\theta_b/(3+\theta_b) = 0.9215$
$\mathbb{V} = (0, 10, 20)$,
$\mathsf{Pr}(V) = (1/10, 4/5, 1/10)$, and

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{40049}{49000}$	$\frac{4}{49}$	0
S_2	$\frac{8951}{49000}$	$\frac{9}{490}$	$\frac{243}{12250}$
S_3	0	$\frac{9}{10}$	$\frac{12007}{12250}$

(40)

For Figures 2 and 3 we use

$\kappa_b/(3+\kappa_b) = 0.9496$
$\theta_b/(3+\theta_b) = 0.4294$
$\mathbb{V} = (0, 10, 20)$,
$\mathsf{Pr}(V) = (1/100, 98/100, 1/100)$, and

$\mathsf{Pr}(S\|V)$	V_1	V_2	V_3
S_1	$\frac{601}{1000}$	$\frac{27}{100}$	0
S_2	$\frac{399}{1000}$	$\frac{18}{100}$	$\frac{245}{1000}$
S_3	0	$\frac{55}{100}$	$\frac{755}{1000}$

(41)

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