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No. 5120

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TIME: WHEN ARE INFORMATIONAL  
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*FINANCIAL ECONOMICS, INDUSTRIAL  
ORGANIZATION and INTERNATIONAL  
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Discussion Paper No. 5120  
July 2005

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July 2005

## **ABSTRACT**

### **Social Learning in Continuous Time: When are Informational Cascades More Likely to be Inefficient?\***

In an observational learning environment rational agents may mimic the actions of the predecessors even when their own signal suggests the opposite. In case early movers' signals happen to be incorrect society may settle on a common inefficient action, resulting in an inefficient informational cascade. This paper models observational learning in continuous time with endogenous timing of moves. This is the first paper with homogenous access to information that gives an analytical approximation for the probability of inefficient cascades. This permits the analysis of comparative statics results. In contrast to the general impression in the literature, the effect of an increase in signal quality on the likelihood of an inefficient cascade is shown to be non-monotonic. If agents do not have strong priors, an increase in signal quality may lead to a higher probability of inefficient herding. The analysis also suggests that markets with quick response to investment decisions, such as financial markets, are more prone to inefficient collapses.

Keywords: comparative statics, herd manipulation and herding

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\*I would like to thank Rabah Amir, Christophe Chamley and Ivan Pastine for their invaluable assistance. This paper has been presented at the 2004 North American Summer Meetings of the Econometric Society. I would like to thank the participants for their helpful comments. Responsibility for errors remains my own.

Submitted 16 June 2005

## 1. Introduction

Individuals with limited information often observe other people's actions before making their own decisions. The predecessors' actions tend to contain information about their private signals concerning the state of nature. In the social learning process the individual aggregates his own private signal with information collected from his observations of others' actions. This process may lead to herd behavior where the agent follows the crowd even when his private information suggests the opposite.<sup>1</sup> In case early movers' signals happen to be incorrect, agents may settle on a common inefficient action, resulting in an inefficient informational cascade. Gale (1996), Chamley (2004) and Bikhchandani, Hirshleifer and Welch (1998) give extensive lists of empirical phenomena that herding behavior may explain in both financial and real markets.<sup>2</sup> Examples include R&D investment decisions, analysts' recommendation of stocks, bank runs and managers decisions to pay dividends. It is often argued that conformist behavior in financial and real markets may lead to sudden booms and crashes.

In seminal herding models by Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992) and Welch (1992) there is an exogenously determined sequence in the moves. Chamley and Gale (1994) extend the literature to allow for endogenous timing of moves. Each agent has an incentive to wait in order to observe the actions of other players. However if everyone were to wait, the agent would rather move early in order to avoid costs of delay. Hence the timing decision is strategic. Chamley and Gale prove the existence of herd behavior in an endogenous-timing framework and characterize the equilibrium. There is a vast literature that examines the conditions under which informational cascades occur and analyzes the speed of cascade formation and the fragility of cascades.<sup>3</sup> There are however no studies that focus on the factors that affect the likelihood of erroneous mass behaviour. This paper studies the factors that

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<sup>1</sup>An informational cascade implies herding behavior. See Smith and Sørensen (2000) for the distinction.

<sup>2</sup>There are a wide variety of markets where herding may arise. For instance, see Scharfstein and Stein(1990), Welch(1992), Devenow and Welch (1996), Avery and Zemsky(1998), Welch (2000), Chari and Kehoe (2003), and Chamley (2003) for analysis of herd behavior in financial markets, Neeman and Orosel (1999) for analysis in auctions, Morton and Williams (1999) for herding in a political economy framework and Bose, Orosel and Vesterlund (2002), Choi, Dassiou and Gettings (2000) and Kennedy (2002) for herding in Industrial Organization frameworks.

<sup>3</sup>See Chamley (2004) for a review of theoretical advances in the study of informational cascades.

influence the likelihood of inefficient cascades, either when there is an investment boom even though the true value of the project is low (an inefficient positive cascade), or when there is an investment collapse even though the true value is high (an inefficient negative cascade). The question the paper analyzes is not the likelihood of cascade formation but the likelihood that the cascade outcome is inefficient.

The analysis of factors that influence the likelihood of inefficient herding – the discount rate, the signal quality, the prior, the expected value of the investment project – may be important for manipulating the outcome of the social learning game. For instance, the Securities Act of 1933 and the Securities Exchange Act of 1934 were formed in hopes of preventing catastrophic crashes like Black Thursday in 1929. Among other regulations, the acts require that investors receive financial and other significant information concerning securities being offered for public sale. We would like to be able to analyse the implications of an improvement in accounting standards (signal quality) on the probability of an inefficient crash. Likewise, we would like to know the effect on the probability of inefficient collapses of an improvement in trading technology. Other than the directional effects of the factors that influence the probability of inefficient herding, the magnitude of the effect may be of importance as well. For instance, in an IPO decision, reducing the price decreases the probability of collapse where the IPO fails. But we would like to know what the optimal price would be. So far the literature lacks a framework to analyse comparative statics results for all key variables.<sup>4</sup> This paper aims to fill this gap. This is the first paper with homogenous access to information that gives an analytical approximation for the probability of inefficient cascades.<sup>5</sup>

We find that the probability of an inefficient negative cascade goes up as agents get more patient. Agents with a low rate of time preference are more inclined to wait to collect more information about the true value of the project. This conservative attitude makes inefficient negative cascades more likely. In Bikhchandani, Hirshleifer and Welch (1992) simulation results on the effect of signal quality on inefficient

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<sup>4</sup>In exogenous-timing herding models calculation of the probability of an inefficient cascade requires the use of Markov Chains and the problem is essentially the Gambler's Ruin problem which does not have a closed-form solution. Therefore comparative statics of exogenous-timing models have had to rely primarily on simulation analysis.

<sup>5</sup>In our paper, in equilibrium agents follow a critical mass. In a model with heterogenous signal quality Zhang's (1997) result is that all agents imitate the one leader with the most precise signal. The probability of an inefficient cascade is then just equal to the probability of the most precise signal being incorrect.

cascade probabilities are monotonic. As signal quality goes up, inefficient negative and positive cascade probabilities go down. We find that this not the case in general. In our framework, the effect of an increase in the signal quality on the likelihood of an inefficient cascade is shown to be nonmonotonic. While an increase in the signal quality does unambiguously improve expected welfare, if agents do not have strong priors, an increase in the signal quality may lead to a higher probability of inefficient herding.

Our paper adapts the Chamley and Gale (1994) endogenous-timing information-revelation model of investment. The equilibrium is in the same spirit as in Chamley and Gale. However, our trading technology allows our agents to observe others' action in continuous time. The merit of a continuous-time framework is that it enables us to approximate a closed form solution to the probability of inefficient cascades. The learning process can be approximated by a series of Brownian Motions and the familiar boundary crossing probabilities are employed to find the probabilities of inefficient positive and negative cascades.

## **2. Model Overview**

In the Chamley-Gale model the higher the value of the project the more people there are with an option to invest. The project value and number of people with an option are however unknown to the agents. Each agent with an investment option faces a tradeoff between investing and waiting. If the agent invests, he collects the undiscounted payoff but faces the risk of making a loss in case the true value of the project is low. If the agent waits, he collects only the discounted payoff but he can learn from others' actions. In equilibrium the agent is just indifferent and randomizes between waiting and investing. Therefore, the rate of investment each period is stochastic. The agent tries to deduce from the rate of investment each period the number of people with an option and hence the value of the project. If the expected value of the project turns negative the agent strictly prefers to wait as does everyone else and so the game ends with an investment collapse. If the expected value of the project becomes so high that agents strictly prefer to invest, the game ends with a sudden investment boom. Since the learning process is stochastic, the outcome may be inefficient. Even when the true value of the project is high (low), there might be an investment collapse (boom).

Here we adapt a special case of the model in which there are only two possible values of the project allowing agents' beliefs to be summarized in a single variable, the probability that the project value is high. It will be shown that a monotonic transformation of the subjective probability yields a

learning process characterized by a simple Lévy process:  $w_t = w_{t-1} - a + bk_t$  where  $k_t$  is the stochastic number of buyers observed in the period and it is given by the Poisson approximation to the binomial distribution.  $w_t$  denotes the transformed subjective probability. The problem is to determine the probability that  $w_t$  crosses the upper bound (where the agent strictly prefers to invest) before the lower bound (where the agent strictly prefers to wait) resulting in an investment boom and the probability that  $w_t$  crosses the lower bound before the upper bound, resulting in an investment collapse. This is a mathematical problem that arises in a number of contexts from psychology to queuing theory and hence has been extensively studied. Nevertheless, despite its seeming simplicity no closed-form solution has been found. Hence we depart from the Chamley and Gale framework and move to a continuous-time learning setting. A natural continuous-time analog to the above stated problem which has known boundary crossing probabilities is Brownian motion.

We have an agency which takes state-contingent orders in discrete time and processes them in continuous time. Invest orders are taken at the beginning of each time period and then implemented randomly during the period. The payoffs are received at the end of the period. If the expected level of investment were constant in all periods, this would result in a continuous-time process for the transformed beliefs that is identical to the evolution of queue lengths in a standard M/D/1 queue. The standard approximation for this is Brownian motion. Our learning process is then a series of one period Brownian motions. While the approximation to Brownian motion is standard in queuing theory, in fact it is a good approximation only when the expected number of buyers in a period is large. To ensure that this is the case we add noise to the system by introducing copycat traders.

In order to make use of the boundary crossing probabilities for Brownian Motion, we need to ensure that in equilibrium the learning process stops as soon as one of the boundaries is crossed. This is done by allowing agents to make their invest and wait orders (placed in discrete time) contingent on the continuously evolving flow of order processing. It will be shown that by introducing a capacity limit to the number of contingent orders the agency can accommodate we are able to keep the same basic incentives as in Chamley and Gale while ensuring that information transmission stops as soon as one of the boundaries is hit.

### 3. Framework

The true value of the investment project is denoted by  $V \in \{V^H, V^L\}$  where  $V^H > 0$  and  $V^L < 0$ .  $V = V^H$  with prior probability  $q^* \in (0, 1)$ . The number of rational players  $rN$  is unboundedly large,  $r \rightarrow \infty$ .<sup>6</sup> Players are risk neutral. They are ex-ante identical. However, only  $rn$  of them receive an investment option. Options are identical and indivisible. If the true value of the project is high, there are more agents with an investment option:

$$n = \begin{cases} n^H & \text{when } V = V^H \\ n^L & \text{when } V = V^L \end{cases} \quad (1)$$

Denote  $\psi \equiv \frac{n^L}{n^H} < 1$ .  $\psi$  is a measure for signal quality. The further apart  $n^L$  and  $n^H$ , the more information the agent has about the true value of the project from the mere fact that he has observed the investment option in the first place. If  $n^L$  were equal to  $n^H$ , the agent who receives the investment option would not update his prior. The agent will update his prior belief more heavily as  $\psi$  goes down.<sup>7</sup>

Each agent with an investment option can give an invest order at any date  $T=0, 1, 2, \dots$  of his choice. Payoffs from the exercised options are received at the end of the time period.  $\delta \in (0, 1)$  is the common discount factor. Investment is irreversible. If the player never invests the payoff is 0. Whether or not the player has an option is private information. Only if the option is exercised is information revealed. Each player with an option chooses to either invest now or delay. When making their decisions, players can observe the history of other players' investments.

So far this is a special case of Chamley and Gale. Chamley and Gale assume that the number of people with an option is stochastic but it is more likely to be high when the true value of the project is high. However, here the value of the project is either high or low and there is a one-to-one mapping between  $V$  and  $n$ . The restriction to only two possible project values will allow us to summarize agents'

<sup>6</sup>This paper adapts the *r-fold replica* game of Chamley and Gale (1994), Section 6.

<sup>7</sup>In order to compare this signal quality measure to the signal quality measure in Banerjee(1992) and Bikhchandani, Hirshleifer and Welch (1992), suppose that each of the  $rN$  potential investors receive a signal about the quality of the project. The signal is correct with probability  $p > 1/2$ . If  $V = V^H$  then  $prN = rn^H$  get a positive signal. If  $V = V^L$  then  $(1-p)rN = rn^L$  get the positive signal. Then  $1/\psi = p/(1-p)$ .



beliefs about the true state of nature at time  $t$  via a single variable: the probability that the project value is high. This mapping will prove to be very convenient in eventually formulating the learning process in a linear fashion.

### 3.1. Copycat Traders

In this paper, we will also include copycat traders. There are  $\Phi = \phi n^H$  people who are randomly assigned to this market and they simply imitate the probability of investment of rational traders with an investment option. The copycats are not needed for the equilibrium of model derived here. However it will be shown that the  $\Phi$  people add noise to the information to be collected from the market. So they make it harder for the market participants to deduce the true value of the project from the observed purchases—resulting in a higher expected number of invest orders per period. In equilibrium, this will not alter the amount of information transmitted by observational learning, but it leads to a larger number of buyers required in equilibrium to transmit the same amount of information. Hence existence of copycats will be shown to improve the quality of the approximation to a tractable learning process.

### 3.2. Trading Technology

Agents place discrete-time state-contingent orders which get processed in continuous time. Players place their orders at the beginning of each period. Orders are processed randomly during the period. The exact time that an individual order is processed is distributed uniformly in the period. Payoffs on all orders processed in a period are received at the end of the period. Since information on others' actions will be arriving during the period, learning is continuous. Players are permitted to make their orders (both invest and wait orders) contingent on the flow of information. Each invest order comes with a state-contingent wait order. Investment cannot be reversed in case the invest order is already processed. Likewise each wait order comes with a state-contingent invest order.

We will eventually approximate the agents' learning process as a series of one-period Brownian Motions with absorbing boundaries. We can make use of the familiar boundary crossing probabilities only if the equilibrium is such that the game stops the first time the learning process hits either of the bounds. Therefore, we limit the capacity of the agency to process contingent orders. During the interval  $[T, T+1)$ , if the state-contingent wait order is triggered then at most  $M$  invest orders are canceled.  $M$  is a very large but finite number. During the interval  $[T, T+1)$ , if the state of the state-contingent invest order is triggered,

then at most  $M-\kappa$  of the newly triggered invest orders are processed randomly during the remainder of the period where  $\kappa$  is the number of invest orders placed at the beginning of the period.<sup>8</sup> After the description of the equilibrium strategies, the advantage of allowing for state-contingent orders and limiting the maximum capacity of the processing agency will become clearer.

#### 4. Equilibrium

The focus is restricted to symmetric Markov Perfect Equilibria. The only payoff-relevant state variable is the subjective probability at time  $t$  that the project value is high, denoted by  $q_t$ .<sup>9</sup> Before describing the equilibrium strategies, let us first introduce some critical values.

##### 4.1. Critical Values

The prior probability that  $V=V^H$  is  $q^*$ . Since orders are processed in continuous time,  $q_t$  evolves in continuous time. The index of time for discrete-time decision nodes is denoted by  $T$ . While  $t \in \mathbb{R}^+$ , the index  $T$  belongs to the set of nonnegative integers. So, at time nodes when  $t=T$ ,  $q_t=q_T$ . Bayes' rule is assumed to describe the agents' method of updating the probabilities. At the beginning of the game, the probability that the project has a high value  $q_{T=0}$ , conditional on having received an investment opportunity, is given by:

$$q_0 = \frac{\frac{rn^H}{rN}q^*}{\frac{rn^H}{rN}q^* + \frac{rn^L}{rN}(1-q^*)} \quad (2)$$

Since  $\psi \equiv \frac{n^L}{n^H}$ , (2) can be rewritten as,

$$q_0 = \frac{q^*}{q^* + \psi(1-q^*)} \quad (3)$$

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<sup>8</sup>The capacity to accommodate state-contingent wait orders  $M$  doesn't need to be the same  $M$  as in the capacity to accommodate state-contingent invest orders  $M-\kappa$ . They just need to be very large finite numbers. One could make the limit on contingencies dependent on how busy the agency will be in the remainder of the period: A limit of  $Mdt$  where  $dt$  is the amount of time remaining in the period. This would necessitate keeping track of more notation but would not change the results.

<sup>9</sup>Chamley and Gale look at the less restrictive set of Perfect Bayesian Equilibria. The Markov restriction allows for a simpler derivation. The resulting equilibria are still Perfect Bayesian Equilibria, see Maskin and Tirole (1988). However the restriction implies that we will not be able to rule out the existence of other non-Markov equilibria.

The game is of interest if initially the expected value of the project is positive:

$$q_0 V^H + (1 - q_0) V^L > 0 \quad (4)$$

Otherwise each agent would strictly prefer to wait and the game would end immediately with an investment collapse.

It will be useful to introduce two critical values for the subjective probability. Define  $\underline{q}$  as the probability where the expected value of the project is zero:

$$\underline{q} V^H + (1 - \underline{q}) V^L = 0 \quad (5)$$

If  $q_t$  falls below  $\underline{q}$ , the agent strictly prefers to wait since the expected value of investment is negative. Since everyone who has not yet invested is identical they all prefer to wait and investment stops for good.

Define  $\bar{q}$  as the probability where the agent is just indifferent between investing now and waiting even though information about the true value of the project is to be fully revealed with certainty next period:

$$\bar{q} V^H + (1 - \bar{q}) V^L = \delta \bar{q} V^H \quad (6)$$

If  $q_t$  rises above  $\bar{q}$ , the agent will strictly prefer to invest now. And so will all the identical players and the game ends with an investment boom where all players with an option invest. The game will be said to be *active* when  $\underline{q} < q_t < \bar{q}$ .

#### 4.2. Endogenous Information Revelation and Learning

Let  $\lambda_T$  denote the probability that a player who has not yet exercised his option puts in an invest order at the decision node T. In the *active* phase of the game, it must be that  $0 < \lambda_T < 1$ . Assume for a moment that an agent expects all people with an investment opportunity to invest this period. Then he would strictly prefer to wait to be able to learn the value of the project for sure. But so would everyone else. Hence  $\lambda_T \neq 1$ . Let us now consider the case where he expects nobody else to invest this period. Given that he expects no new information to arrive this period, he wouldn't prefer to wait. Next period he would have the same expected value of the game as in this period, but discounted. Hence he would strictly prefer to invest now, so  $\lambda_T \neq 0$ , by contradiction. Hence in the *active* phase of the game, there can be no pure-strategy equilibrium,  $0 < \lambda_T < 1$ . Players are just indifferent between waiting and investing now.

Notice that  $\lambda_T$  is the endogenous rate of information flow. If  $\lambda_T$  were zero, no information would be revealed. If  $\lambda_T$  were equal to one, the number of people who invest would fully reveal information about the value of the project.

In equilibrium, the history of the game will affect players' strategies only through the subjective probability  $q_t$  and therefore  $\lambda_T = \lambda(q_T)$ . While players' actions depend on the publicly observed history of the game, history could be payoff relevant for two reasons: i) It influences players' beliefs about the probability that the project has a high value ii) As more and more people invest, the number of potential investors with unexercised options goes down and hence history could potentially alter the future flow of information. However the second argument cannot apply in equilibrium. Notice that the expected number of invest orders, either  $\lambda_T(\psi+\phi)rn^H$  when  $V=V^L$  or  $\lambda_T(1+\phi)rn^H$  when  $V=V^H$ , must be finite. If it were infinite, the observation of the rate of investment in one period would reveal the true value of  $n$ , and hence  $V$  by the law of large numbers. In such a case all players would strictly prefer to wait, implying  $\lambda_T=0$ , a contradiction. As  $r \rightarrow \infty$ , the population of potential investors is very large so players are essentially sampling with replacement. For any finite number of exercised options there is still an infinitely large number of potential investors. Hence the history of the game is payoff relevant only because it influences the agents' belief that the project has a high value. In order to save on notation, we will denote  $\lambda(q_T)$  as  $\lambda_T$ .

Since agents randomize with  $0 < \lambda_T < 1$ , the level of investment each period is stochastic. As  $r \rightarrow \infty$ , the number of people putting in invest orders at a decision node is given by the Poisson approximation to the binomial distribution. The parameter of the Poisson distribution is  $(rn\lambda_T + \phi\lambda_T rn^H)$ , the mean number of invest orders by the investors  $\lambda_T rn$  plus the expected number of invest orders by the copycats  $\phi\lambda_T rn^H$ . During the period these investors' orders will be processed randomly. Hence the time between completed orders has the exponential distribution and the number of orders processed in any time interval  $\Delta t \in (0,1)$  is distributed Poisson with parameter  $(rn\lambda_T + \phi\lambda_T rn^H)\Delta t$ .

Take any  $x \in [0,1)$  and any  $\Delta t \in (0,1-x)$ . When  $V=V^H$ , the probability that  $k$  players' investments are processed during the time  $\Delta t$  given  $\lambda_T$  is denoted by  $f^H(k, \lambda_T, \Delta t)$ :

$$f^H(k; \lambda_T, \Delta t) = \begin{cases} \frac{e^{-(1+\phi)\lambda_T rn^H \Delta t} [(1+\phi)\lambda_T rn^H \Delta t]^k}{k!} & \text{for } k \in \mathbb{Z} \cap [0, (1+\phi)rn^H] \\ 0 & \text{elsewhere} \end{cases} \quad (7)$$

and the p.d.f. is denoted by  $f^L(k, \lambda_T, \Delta t)$  when  $V=V^L$ :

$$f^L(k; \lambda_T, \Delta t) = \begin{cases} \frac{e^{-(\psi+\phi)\lambda_T r n^H \Delta t} [(\psi+\phi)\lambda_T r n^H \Delta t]^k}{k!} & \text{for } k \in \mathbb{Z} \cap [0, (\psi+\phi)r n^H] \\ 0 & \text{elsewhere} \end{cases} \quad (8)$$

If  $n^L$  were equal to  $n^H$ , then  $\psi=1$  and the two probability density functions would collapse together. In such an extreme case the quality of the information  $k$  contains would be nil and the observation of the rate of investment would not reveal any information.

The agent tries to deduce from the number of people who invest in each period which distribution the observation comes from. Define  $k_{\Delta t}$  as the number of invest orders processed during the time  $\Delta t$ . Assuming no contingencies are triggered Bayesian learning implies that at time  $T+x+\Delta t$ , when the agent observes  $k_{\Delta t}$  people investing, the subjective probability will evolve following:

$$q_{T+x+\Delta t} = \frac{q_{T+x} f^H(k_{\Delta t}; \lambda_T, \Delta t)}{q_{T+x} f^H(k_{\Delta t}; \lambda_T, \Delta t) + (1 - q_{T+x}) f^L(k_{\Delta t}; \lambda_T, \Delta t)} \quad (9)$$

The possibility of contingencies being triggered is discussed below.

### 4.3. Equilibrium Strategies

Let us first assume that the institutional environment restricts the agents to only use  $\underline{q}$  and  $\bar{q}$  as their triggers for the contingent orders. In Appendix B, this assumption is relaxed. The equilibrium of the game with any finite set of possible trigger points which contains  $\bar{q}$  and  $\underline{q}$  is shown to yield the same cascade probabilities as this baseline model.

PROPOSITION 1: *The following equilibrium strategies support a symmetric equilibrium:*

- a) *If the subjective probability is sufficiently low  $q_T \leq \underline{q}$ , put in a wait order with a state-contingent invest order. If in the time interval  $[T, T+1)$ ,  $q_t \geq \bar{q}$ , the state-contingent invest order is triggered.*
- b) *If the subjective probability is sufficiently high  $q_T \geq \bar{q}$ , put in an invest order with a state-contingent wait order. If in the time interval  $[T, T+1)$ ,  $q_t \leq \underline{q}$ , the state-contingent wait order is triggered.*
- c) *If the subjective probability is  $\underline{q} < q_T < \bar{q}$ , with probability  $\lambda_T$ , put in an invest order with a state-contingent wait order. If in the time interval  $[T, T+1)$ ,  $q_t \leq \underline{q}$ , the state-contingent wait order is triggered. With probability  $(1-\lambda_T)$  put in a wait order with a state-contingent invest order. If in the time interval  $[T, T+1)$ ,  $q_t \geq \bar{q}$  the state-contingent invest order is triggered.*

PROOF: a) By equation (5), when the subjective probability is  $\underline{q}$ , the expected value of the project is just equal to zero. All agents prefer to wait when  $q_T \leq \underline{q}$ . Hence the state  $q_T \leq \underline{q}$  is absorbing. Investment stops for good. Information transmission terminates. There is no possible deviation from the equilibrium strategy that would make the agent better off. Off the equilibrium path, if in the time interval  $[T, T+1)$  new information arriving leads to an updated belief  $q_t \geq \bar{q}$ , the agent prefers to invest and the state-contingent invest order is triggered. Since  $r \rightarrow \infty$ , and only  $M-\kappa$  invest orders can be processed, the probability of an individual agent's state-contingent invest order being processed is zero.

b) When  $q_T \geq \bar{q}$ , by equation (6) the agent strictly prefers to invest now even if this period the true value of the project is to be revealed for sure. When  $q_T \geq \bar{q}$ , all agents with an option give invest orders. Since  $r \rightarrow \infty$ , during the time interval  $[T, T+1)$  the true value of  $n$  and hence  $V$  is revealed at once. If  $V=V^H$  the subjective probability would remain above  $\bar{q}$ . If  $V=V^L$ , the subjective probability would drop below  $\underline{q}$ . All agents' state contingent wait orders would be triggered at once but only  $M$  of them would be able to stop investment. The game would end with all investing except of those lucky  $M$ . Therefore,  $q_T \geq \bar{q}$  is an absorbing state as well.

When  $q_T \geq \bar{q}$  the individual agent could consider giving a wait order with a state contingent invest order. If all follow the above described equilibrium strategy, information about the true value of the project would be revealed at once, and the agent's state contingent invest order would be triggered at once. However his order would be not served in the period, since at most  $M-\kappa$  of the newly triggered orders are processed during the period. In equilibrium, when  $q_T \geq \bar{q}$  all give invest orders, since  $r \rightarrow \infty$  so does  $\kappa$ . Hence

no newly triggered invest orders can be processed. Therefore a deviation would lead to one period of discounting. By equation (6) the agent would strictly prefer to follow the equilibrium strategy.

c) If  $\underline{q} < q_T < \bar{q}$ , the expected value from investment is positive but the agent will also consider waiting in order to learn about the true value of the project. In equilibrium the agent is just indifferent between investing now and waiting. See the beginning of Section 4.2 for the discussion of the non-existence of pure-strategy equilibrium.

i) The agent with an investment option who has not yet exercised his option will put an invest order at time  $T$  with probability  $\lambda_T$ . If however in the time interval  $[T, T+1)$ ,  $q_t$  falls below  $\underline{q}$ , the agent would prefer to wait by equation (5). All unprocessed invest orders would convert into wait orders. Since  $M$  is a very large number, investment would stop for good.

ii) The agent with an investment option who has not yet exercised her option will put a wait order at time  $T$  with probability  $(1-\lambda_T)$ . If however in the time interval  $[T, T+1)$ ,  $q_t$  rises above  $\bar{q}$  the agent would prefer to invest.  $M$  is very large but finite.  $M-\kappa$  newly arrived invest orders would be processed this period. All the rest would be processed next period. At time  $T$ , each agent realizes that there is zero probability that his invest order would be processed if the state is triggered.

□

At this point the advantage of allowing for contingent orders should be clear: In the active phase of the game, the subjective probability evolves in continuous time within the bounds  $\underline{q}$  and  $\bar{q}$ . Because we let potential investors put in contingent orders, the game stops the first time  $q_t$  hits either of the bounds. Once either of the bounds is hit, information transmission stops. Without contingencies, within the time interval  $[T, T+1)$ , the subjective probability could potentially cross one the bounds and then bounce back depending on the flow of the invest order processing since the exact timing of the processing of the individual order is a random variable. In the active phase of the game, if the process hits  $\underline{q}$ , state-contingent wait orders are triggered and they get served since  $M$  is a very large number. Therefore information flow would stop. If the process hits  $\bar{q}$ , all state-contingent invest orders are triggered. If the processing agency were not to have any capacity constraint, the true value of the project would be revealed at once and potentially the subjective probability could drop below  $\underline{q}$ . However the agency can process only  $M-\kappa$  newly triggered invest orders. So, no matter whether the true  $n$  is  $n^H$  or  $n^L$  the same number of invest orders will be processed. Therefore in the remaining time, the agent cannot gain further information

about the true value of the project. Information flow stops the moment the subjective probability hits the upper bound.

## 5. Information Cascades

Agents beliefs about the true value of the project evolve as a result of observational learning. If the subjective probability hits  $\underline{q}$  before  $\bar{q}$ , the game ends with an investment collapse. If the subjective probability hits  $\bar{q}$  before  $\underline{q}$ , the game ends with an investment boom. We are particularly interested in the probability of inefficient cascades. The measures of interest are then the probability that the process hits  $\underline{q}$  before  $\bar{q}$  when  $V = V^H$ , and the probability that the process hits  $\bar{q}$  before  $\underline{q}$  when  $V = V^L$ . The first is an inefficient negative cascade and the latter is an inefficient positive cascade.

While the paper will discuss both types of inefficient outcomes, notice that only inefficient negative cascades would be categorized as inefficient herding. Here agents that receive an investment option would invest if learning were not permitted (see Eq 4). Since by definition herding is acting against one's own signal, we can talk about inefficient herding only when the crowd chooses not to invest.

### 5.1. Transformation

In order to obtain the boundary crossing probabilities, we need to transform the problem into an equivalent problem that is tractable. Probabilities evolve following (9), substitute  $f^H(k; \lambda_T, \Delta t)$  and  $f^L(k; \lambda_T, \Delta t)$  from (7) and (8) into (9). Cancel out  $k$  factorial from the numerator and denominator. Take the inverse of both the left and right hand side of the equality and subtract one from each side. Now plugging in  $\psi$  for  $\frac{n^L}{n^H}$  yields,

$$\frac{1 - q_{t+\Delta t}}{q_{t+\Delta t}} = \frac{1 - q_t}{q_t} e^{(1-\psi)\lambda_T r n^H \Delta t} \left( \frac{\psi + \phi}{1 + \phi} \right)^{k_{\Delta t}} \quad (10)$$

Taking the natural logarithm and multiplying by minus one yields:

$$-\ln\left(\frac{1 - q_{t+\Delta t}}{q_{t+\Delta t}}\right) = -\ln\left(\frac{1 - q_t}{q_t}\right) - (1 - \psi)\lambda_T r n^H \Delta t - k_{\Delta t} \ln\left(\frac{\psi + \phi}{1 + \phi}\right) \quad (11)$$



where  $k_{\Delta t}$  is distributed Poisson with the parameter  $\lambda_T(1 + \phi)rn^H \Delta t$  when the true value of the project is high and it is distributed Poisson with the parameter  $\lambda_T(\psi + \phi)rn^H \Delta t$  when the true value of the project is low. Define a transformation  $w_t$  as:

$$w_t \equiv -\ln\left(\frac{1 - q_t}{q_t}\right) \quad (12)$$

Notice that  $w_t$  is an increasing monotonic transformation of  $q_t$ . We can simply rewrite (11) as:

$$w_{t+\Delta t} = (\psi - 1)\lambda_T rn^H \Delta t + w_t - \ln\left(\frac{\psi + \phi}{1 + \phi}\right) k_{\Delta t} \quad (13)$$

The transformed subjective probabilities evolve following (13), where  $k_{\Delta t}$  is investment in  $\Delta t$ .

In this model individual learning follows a stochastic process with independent increments. It is interesting to note that this process is a well known description of individual learning in cognitive psychology. That literature looks, for example, at how people identify objects looking at pictures. In much of the literature individuals are modeled as learning through random sampling. This characterization of the learning process is then used to explain laboratory evidence on individual response times and error rates. The present paper shows that even with fully rational agents group behavior will resemble individual behavior of boundedly rational agents of the type used in cognitive psychology.<sup>10</sup>

The transformation (12) of the lower bound given by (5), of the upper bound given by (6) and of the starting point given by (3) yield :

$$\text{The lower bound: } \underline{q} \Rightarrow \underline{w} \quad \underline{w} = \ln(-V^L) - \ln V^H \quad (14)$$

$$\text{The upper bound: } \bar{q} \Rightarrow \bar{w} \quad \bar{w} = \ln(-V^L) - \ln V^H - \ln(1 - \delta) \quad (15)$$

$$\text{The starting point: } q_0 \Rightarrow w_0 \quad w_0 = \ln q^* - \ln(1 - q^*) - \ln \psi \quad (16)$$

Since initially the expected value of the project is positive (4),  $\underline{w} < w_0$ . And  $w_0 < \bar{w}$  examining (6) and (3) together.

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<sup>10</sup>See Luce (1986) for an introduction to this literature.

## 5.2. Boundary Crossing Probabilities with constant $\lambda$

Transformed subjective probabilities follow equation (13) where the error term  $k_{\Delta t}$  is distributed Poisson with the parameter  $(rn\lambda_T + \phi\lambda_T rn^H)\Delta t$ . Both the mean and the variance of the process depend on  $\lambda_T$  and hence they depend on the history of the game. They are not constant.

Now we are going to examine a different process. In this modified problem we will examine the processes described by equations (13) and (7) and (8) yet with a constant  $\lambda \in (0,1)$ , implying a constant Poisson parameter. In Section 5.3, we will prove that the process with the endogenously determined non-constant  $\lambda$  will yield identical boundary crossing probabilities as in the modified problem with fixed  $\lambda$ .

With a constant  $\lambda$ , equation (13) implies that  $w_t$  evolves in the same way as the queue length in a standard M/D/1 queue. An M/D/1 queue has exponential arrivals, so the distribution of new customers over an interval  $\Delta t$  is Poisson, and one server who takes a deterministic amount of time to serve a customer. Here we will make use of the standard results for the queue length for heavy-traffic M/D/1 queues and approximate the evolution of  $w_t$  as a Brownian Motion<sup>11</sup>:

$$w_{t+\Delta t} \approx w_t + k_{\Delta t}^* \quad (17)$$

where  $k_{\Delta t}^*$  is distributed normal with mean  $\mu\Delta t$  and variance  $\sigma^2\Delta t$ :

$$\text{when } V = V^H \quad k_{\Delta t}^* \sim N \left( \begin{array}{l} \mu^H = \left( \psi - 1 - (1 + \phi) \ln \frac{\psi + \phi}{1 + \phi} \right) \lambda rn^H, \\ (\sigma^2)^H = (1 + \phi) \left( \ln \frac{\psi + \phi}{1 + \phi} \right)^2 \lambda rn^H \end{array} \right) \quad (18)$$

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<sup>11</sup>This is known as the ‘‘heavy-traffic diffusion approximation.’’ A heavy-traffic queue is one in which the average arrival rate of customers is close to the capacity of the server. In queuing theory this is important for the approximation because otherwise the queue length would have a mass point at zero length. This issue does not arise here because the game will effectively end when either of the boundaries are hit. Here we have absorbing boundaries rather than the reflective boundary of a queue.

$$\text{when } V = V^L \quad k_{\Delta t}^* \sim N \left( \begin{array}{l} \mu_T^L = \left( \psi - 1 - (\psi + \phi) \ln \frac{\psi + \phi}{1 + \phi} \right) \lambda_T r n^H, \\ (\sigma_T^2)^L = (\psi + \phi) \left( \ln \frac{\psi + \phi}{1 + \phi} \right)^2 \lambda_T r n^H \end{array} \right) \quad (19)$$

$\mu^H > 0$  by Claim A1, and  $\mu^L < 0$  by Claim A2 in the Appendix.

It is important to emphasize the limitations of this approximation. It is a good approximation only when the expected rate of investment is high. This will happen with a high  $\Phi$ . Keeping the rate of information flow  $\lambda$  constant, an increase in  $\Phi$ , leads to a weaker drift. That is, the positive drift  $\mu^H$  declines and the negative drift  $\mu^L$  increases. The higher  $\Phi$  the less informative an observation is about the true value of the project. Both  $(\sigma^2)^H$  and  $(\sigma^2)^L$  go down with an increase in  $\Phi$  keeping  $\lambda$  constant. Each observation has less informational content hence the beliefs don't get updated as much. Therefore keeping  $\lambda$  constant, an increase in  $\Phi$  would make agents strictly prefer to wait. Since in equilibrium agents are just indifferent between investing and waiting, a higher  $\Phi$  must be associated with a higher  $\lambda$ . Moreover as  $q_T \rightarrow \bar{q}$ ,  $\lambda_T r n \rightarrow \infty$ . Hence for a given  $\Delta t$  and  $q_T$ , there is a  $\Phi$  that will yield a high enough Poisson parameter  $(r n \lambda_T + \phi \lambda_T r n^H) \Delta t$  so that the normal approximation is reasonable. For any finite  $\Phi$  there will be a range  $q_T \in (\underline{q}, \tilde{q}]$  where the approximation is not reasonable. Notice however that  $\lambda$  only changes at  $t \in \mathbb{Z}$ , so by increasing  $\Phi$  and hence decreasing  $\tilde{q}$  we can make the probability of ever encountering  $q_T \in (\underline{q}, \tilde{q}]$  small. Brownian Motion also requires that the normal approximation holds as  $\Delta t \rightarrow 0$ . Just as in queuing theory there is no set of parameters where this is the case.

The standard boundary crossing probabilities for Brownian motion yield the following results.<sup>12</sup>

i) *Probability of hitting  $\underline{w}$  before  $\bar{w}$  when  $V = V^H$ :*

$$\text{Prob}(\underline{w} \text{ before } \bar{w} | V = V^H) = 1 - \left[ \frac{e^{\frac{-2\mu^H \underline{w}}{(\sigma^2)^H}} - e^{\frac{-2\mu^H \bar{w}_0}{(\sigma^2)^H}}}{e^{\frac{-2\mu^H \underline{w}}{(\sigma^2)^H}} - e^{\frac{-2\mu^H \bar{w}}{(\sigma^2)^H}}} \right] \quad (20)$$

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<sup>12</sup>See for example Karlin and Taylor (1975) for the derivation.

Multiply the numerator and the denominator of (20) by  $\exp\left(\frac{2\mu^H \underline{w}}{(\sigma^2)^H}\right)$ . Plug in Equations (18), (14), (15) and (16) for  $\underline{w}$ ,  $\bar{w}$ ,  $w_0$ ,  $\mu^H$  and  $(\sigma^2)^H$ :

$$Prob(\underline{w} \text{ before } \bar{w} \mid V = V^H) = 1 - \left( \frac{1 - e^{2\varphi^H [\ln(1-q^*) - \ln q^* + \ln \psi - \ln V^H + \ln(-V^L)]}}{1 - e^{2\varphi^H \ln(1-\delta)}} \right) \quad (21)$$

where  $\varphi^H \equiv \frac{\mu^H}{(\sigma^2)^H}$ . By Claim A1 in Appendix A,  $\mu^H > 0$ . So  $\varphi^H > 0$ .

ii) Probability of hitting  $\underline{w}$  before  $\bar{w}$  when  $V = V^L$ :

$$Prob(\bar{w} \text{ before } \underline{w} \mid V = V^L) = \left( e^{\frac{-2\mu^L \underline{w}}{(\sigma^2)^L}} - e^{\frac{-2\mu^L w_0}{(\sigma^2)^L}} \right) / \left( e^{\frac{-2\mu^L \underline{w}}{(\sigma^2)^L}} - e^{\frac{-2\mu^L \bar{w}}{(\sigma^2)^L}} \right) \quad (22)$$

Plug in (19), (14), (15), (16) into (22):

$$Prob(\bar{w} \text{ before } \underline{w} \mid V = V^L) = \frac{1 - e^{2\varphi^L [\ln(1-q^*) - \ln q^* + \ln \psi - \ln(V^H) + \ln(-V^L)]}}{1 - e^{2\varphi^L \ln(1-\delta)}} \quad (23)$$

where  $\varphi^L \equiv \frac{\mu^L}{(\sigma^2)^L}$ . By Claim A2 in Appendix A,  $\mu^L < 0$ . So  $\varphi^L < 0$ .

Before we proceed to the next subsection it is crucial to notice that  $\varphi^H$  and  $\varphi^L$  are independent of  $\lambda$  from equations (18) and (19). While an increase in  $\lambda$  leads to an increase in  $\mu$ , it also leads to an increase in  $\sigma^2$ . And these two effects counterbalance each other in the determination of the boundary crossing probabilities. Hence these probabilities are independent of  $\lambda$ .

### 5.3. Inefficient Cascade Probabilities for the Original Problem

**PROPOSITION 2:** *The boundary crossing probabilities of the original problem are equal to the boundary crossing probabilities found using a Brownian Motion, (21) and (23) of the modified problem.*

**Proof:**

In the actual learning process, the  $\lambda$  is updated at each decision node. The process is a series of one period Brownian Motions. The boundary crossing probabilities for this process can be reconstructed iteratively using Lemma B1 in Appendix B. Starting with the Brownian Motion with absorbing boundaries defined in (14) and (15) and the starting point given by (16), create a process where the parameter  $\lambda$  changes to  $\lambda'$  (which is stochastic) at  $T=1$  and stays constant thereafter. From the Lemma B1, this new process has the same transition probabilities as the original process. Iterating this argument yields the result.

□

Denote the probability of an inefficient negative cascade by  $Prob(INC)$  which is equal to the probability that the process hits  $\underline{q}$  before  $\bar{q}$  when  $V = V^H$ . By Proposition 2,  $Prob(INC)$  is given by equation (21).

$Prob(IPC)$  denotes the probability of an inefficient positive cascade and it is the probability that the process hits  $\bar{q}$  before  $\underline{q}$  when  $V = V^L$ . By Proposition 2,  $Prob(INC)$  is given by equation (23).

**6. Comparative Statics**

*6.1. The Prior Expected Value:*

When there is potentially a lot to gain ( $V^H \uparrow$ ) or little to lose ( $V^L \uparrow$ ), much would be lost in expectation due to discounting while waiting. So the agent would be more prone to investing before he is certain it is a good project (hence  $\bar{q} \downarrow$ ). And the belief about the odds of the project being a high value project does not need to be as high for the agent to strictly prefer to wait (so  $\underline{q} \downarrow$ ). Therefore the probability of hitting the upper bound before hitting the lower bound increases. The likelihood of an inefficient positive cascade goes up and the likelihood of an inefficient negative cascade goes down.<sup>13</sup>

As the prior expected value of the project goes up due to an increase in  $q^*$ , the upper bound  $\bar{q}$  and the lower bound  $\underline{q}$  are unaffected, from equations (6) and (5). Being closer to the upper bound to begin

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<sup>13</sup> In an exogenous-timing herding framework Welch (1992) shows that as the prior expected value from investment goes up, there is a greater chance that the society ends up in a positive cascade since early movers are more likely to invest.

with, the agents require less evidence in their learning process to strictly prefer investment. Hence they are more likely to invest when the prior improves.

**PROPOSITION 3:** *As the prior expected value increases ( $q^*$  ↑ or  $V^H$  ↑ or  $V^L$  ↑), it becomes more likely that all agents with an option undertake the project. It becomes less likely that there is an investment collapse when the true value is high,  $Prob(INC)$  ↓. It becomes more likely that there is an investment boom when the true value is low,  $Prob(IPC)$  ↑.*

**Proof:** See Appendix C for the proof of Proposition 3.

### 6.2. Discounting:

Discounting doesn't play a role in exogenous timing models. Examination of this issue requires an endogenous-timing model. The agent makes a choice between investing now or later. If the agent waits, he can learn by observing other people's actions, however the payoff gets discounted. All else equal, as people get more patient ( $\delta$  ↑), they will be more willing to wait. Since waiting induces learning, one might be tempted to conclude that higher  $\delta$  would be associated with a smaller probability of an erroneous mass behavior. However this is not the case. In fact the probability of an inefficient negative cascade goes up.

**PROPOSITION 4:** *As agents become more patient ( $\delta$  ↑), it becomes more likely that agents do not undertake the investment project. It becomes more likely that there is an investment collapse,  $Prob(INC)$  ↑, when the true value is high. It becomes less likely that there is an investment boom,  $Prob(IPC)$  ↓, when the true value is low.*

**Proof:**

The probability of an inefficient negative cascade goes up as  $\delta$  goes up,

$$\frac{dProb(INC)}{d\delta} = (1 - Prob(INC)) \frac{2\varphi^H e^{2\varphi^H \ln(1-\delta)}}{(1-\delta) \left(1 - e^{2\varphi^H \ln(1-\delta)}\right)} > 0 \quad (24)$$

On the other hand, the probability of an inefficient positive cascade goes down as  $\delta$  goes up.

$$\frac{dProb(IPC)}{d\delta} = -Prob(IPC) \frac{2\varphi^L e^{2\varphi^L \ln(1-\delta)}}{(1-\delta) \left(1 - e^{2\varphi^L \ln(1-\delta)}\right)} < 0 \quad (25) \quad \square$$

An increase in  $\delta$  has a two effects. The first effect is through the rate of information flow  $\lambda$ . As agents get more patient, at the ongoing information flow, they would strictly prefer to wait. However  $\lambda=0$  cannot be sustained in equilibrium. So the rate of information flow goes down such that people are just indifferent between waiting and investing.<sup>14</sup>

However this first effect has no influence on the probability of inefficient cascades in our framework. Here is why: When the true value of the project is high  $V=V^H$ , a weaker information flow implies a weaker drift velocity  $\mu^H$  which simply increases the likelihood of a negative cascade. However at the same time the weaker information flow decreases the noise  $(\sigma^2)^H$  in the learning process. Each observation will have a small influence on the updating process. This reduces the likelihood of a negative cascade. And these two opposing effects exactly counterbalance each other.  $\lambda$  cancels out from the probability of inefficient cascade (see equations (21) and (23)). The indirect effect through the information flow is therefore nullified. The spirit of the story is the same for the case when  $V=V^L$ .

The second effect of an increase in  $\delta$  is through the upper bound. A higher  $\delta$  yields a higher upper bound  $\bar{q}$ , leaving the starting point and the lower bound unchanged. Since investors are more patient, they are willing to wait until they are almost certain about the project before they buy. This makes an inefficient negative cascade more likely, and an inefficient positive cascade less likely.

This comparative statics result suggests that financial markets might be more prone to inefficient collapses than the real markets. Once an investment order is given, the payoff can be collected only at the end of the period. Keeping the rate of time preference constant, as the time to process investment decisions increases so does the distance between the time periods in the model, leading to a lower  $\delta$ . In financial

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<sup>14</sup>See Chamley (2004) for a simple two period two agent example. If  $V=V^H$ , both people have an option. If the  $V=V^L$ , only one agent has an option. In equilibrium, the agent if not invested in the first period would invest in the second period only if he observes investment in the first period. When  $\delta$  goes up, the equilibrium probability of investment in the first period goes down, making an investment collapse more likely even if the true value is high.

markets, the administrative and technological systems may be faster to react to agents investment decisions than in real markets. Hence in financial markets the relevant  $\delta$  would be larger than in real markets leading to a higher likelihood of an inefficient collapse.

### 6.3. Quality of Information:

Simulation results in Bikhchandani, Hirshleifer and Welch (1992) show that as signal quality goes up, inefficient negative and positive cascade probabilities goes down. The effect is monotonic. We find that this not the case in general. An increase in the signal quality does of course unambiguously improve expected value of the game.<sup>15</sup> But the probabilities of inefficient cascades may go up or down depending on the parameter values.

To understand the role of signal quality in our framework, first notice that agents who receive an investment option would all undertake the investment if there were no social learning. It is through social learning that the possibility of an investment collapse arises. The increase in the signal quality affects the outcome of the game through two channels; i) *Self-confidence*: It increases the confidence of the agent in his own signal. Keeping the level of signal quality of the rest of the people constant, as the signal quality of the agent goes up, the agent is more likely to undertake the project, hence  $\text{Prob}(\text{INC}) \downarrow$  and  $\text{Prob}(\text{IPC}) \uparrow$ . ii) *Confidence in observational learning*: It increases the confidence of the agent in observational learning since each individual has a high quality signal. Now keeping the signal quality of the agent constant, as the signal quality of the rest of the players increases the agent becomes more likely not to undertake the project, hence  $\text{Prob}(\text{INC}) \uparrow$  and  $\text{Prob}(\text{IPC}) \downarrow$ . These two channels with opposing forces can be examined below.

Let us first examine the probability of an inefficient negative cascade. Equation (20) can be rewritten as:

$$\text{Prob}(\text{INC}) = 1 - \frac{1 - e^{2\varphi^H(\underline{w}-w_0)}}{1 - e^{2\varphi^H(\underline{w}-\bar{w})}} \quad (26)$$

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<sup>15</sup>See equations (3) and (4).



From equations (14) and (15) notice that  $\underline{w}$  and  $\bar{w}$  are independent of  $\psi$ . So,

$$\frac{d \text{Prob}(INC)}{d\psi} = \left( \frac{\partial \text{Prob}(INC)}{\partial w_0} \frac{\partial w_0}{\partial \psi} \right) + \left( \frac{\partial \text{Prob}(INC)}{\partial \varphi^H} \frac{\partial \varphi^H}{\partial \psi} \right) \quad (27)$$

The first term relates to the first channel; self-confidence. It is the effect of  $\psi$  on the probability of an inefficient negative cascade through the agents's belief before the observational learning starts ( $w_0$ ). It is positive by Claim C3 in Appendix C.<sup>16</sup> The second term relates to the second channel; confidence in observational learning. It is negative by Claim C1 and C4 in Appendix C. Therefore there are two forces working in opposite directions.

Let us now examine the probability of an inefficient positive cascade. Rewrite (22) as,

$$\text{Prob}(IPC) = \frac{1 - e^{2\varphi^L(\underline{w}-w_0)}}{1 - e^{2\varphi^L(\underline{w}-\bar{w})}} \quad (28)$$

So,

$$\frac{d \text{Prob}(IPC)}{d\psi} = \left( \frac{\partial \text{Prob}(IPC)}{\partial w_0} \frac{\partial w_0}{\partial \psi} \right) + \left( \frac{\partial \text{Prob}(IPC)}{\partial \varphi^L} \frac{\partial \varphi^L}{\partial \psi} \right) \quad (29)$$

The first term is negative by Claim C5 in Appendix C. It relates to the first channel. The second term relates to the second channel. It is positive by Claim C2 and Claim C6 in Appendix C. Hence there are two opposing effects.

**PROPOSITION 5:** *As the signal quality improves  $\left(\frac{1}{\psi} \uparrow\right)$ , the likelihood of an inefficient positive cascade and the likelihood of an inefficient negative cascade might go up or down depending on the parameter values.*

**Proof:** See Appendix C for the proof of Proposition 5.

Proposition 5 is proven in two steps. The first step proves that  $\frac{d \text{Prob}(INC)}{d\psi} > 0$ , when  $w_0 \rightarrow \underline{w}$  and  $\frac{d \text{Prob}(INC)}{d\psi} < 0$  when  $w_0$  is not too close to either of the bounds. The second step proves that

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<sup>16</sup>This is consistent with the discussion in the previous paragraph since an increase in  $\psi$  represents a deterioration in signal quality.

$\frac{d \text{Prob}(IPC)}{d\psi} < 0$ , when  $w_0 \rightarrow \bar{w}$ .  $\frac{d \text{Prob}(IPC)}{d\psi} > 0$  when  $w_0$  is not too close to either of the bounds. Depending on the initial belief of the agent with an option, the effect of an increase in signal quality through confidence in observational learning can dominate the effect through self-confidence. If the agent does not have a strong initial belief (such that  $w_0$  is not too close to either of the bounds) the social learning channel becomes more important in determining the direction of the effect of signal quality on the outcome of the game. An increase in signal quality can lead to an increase in the probability of inefficient herding. Investment is stochastic by the very nature of the game. Even when the true value of the project is high, it is possible to get a few bad draws in a row. In such a case, when the signal quality is high, the agent will heavily update his beliefs possibly leading to an investment collapse.<sup>17</sup> If however the signal quality were lower, the agent would need to collect more evidence before he strictly preferred to wait.

While our framework has endogenous timing and the model simulated in Bikhchandani, Hirshleifer and Welch assumes exogenous timing, the difference in the comparative statics results does not hinge on the timing assumption. It is the symmetry of the Bikhchandani, Hirshleifer and Welch model that leads to monotonicity. In an exogenous timing model, Pastine and Pastine (2005) show that a simple generalization of the Bikhchandani, Hirshleifer and Welch model, leads to rich simulation results, consistent with the theoretical predictions of this endogenous-timing model.<sup>18</sup>

## 7. Conclusion

Knowledge of the direction and the magnitude of comparative statics results on the probability of a collapse is essential in herd manipulation. For example, the pricing decision of a firm introducing a new technology or the choice of advertising policy (affecting the signal quality) may be some of the key elements in manipulating the outcome of the social learning game. In financial markets, one of the

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<sup>17</sup>On the other hand, naturally if signal quality is high, it would be less likely to have bad draws when the true value is high.

<sup>18</sup>As the signal quality goes up, it is more likely that the first movers pick the correct choice and the confidence in observational learning increases. But this implies that the agent will be more likely to go with the “trustworthy” crowd which is bad news if the early movers happen to have picked the incorrect choice. Also see De Vany and Lee (2001). Nelson (2002) look at a related but different aspect of signal quality. The simulation results indicate that in a changing environment, the *frequency* of herding is non-monotone in signal quality.

objectives of the regulatory body may be to reduce the probability of sudden crashes.<sup>19</sup> Accounting standards affect the signal quality, transaction taxes affect the expected value, the closing and opening times as well as the time of delay for payments of the payoffs affect discounting.

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<sup>19</sup>Avery and Zemsky (1998) show that herd behavior can lead to a significant short-run mispricing of financial assets.

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## APPENDIX

### Appendix A:

**Claim A1:**  $\mu^H > 0$  where  $\mu^H = \left( \psi - 1 - (1 + \phi) \ln \frac{\psi + \phi}{1 + \phi} \right) \lambda r n^H$ .

**Proof:** Define  $f(\psi) = \left( \psi - 1 - (1 + \phi) \ln \frac{\psi + \phi}{1 + \phi} \right)$ . Note that  $f(1) = 0$ .

Since  $f'(\psi) = \frac{\psi - 1}{\psi + \phi} < 0 \Rightarrow f(\psi) > 0$  for  $0 < \psi < 1$ . So,  $\mu^H > 0$ . □

**Claim A2:**  $\mu^L < 0$  where  $\mu^L = \left( \psi - 1 - (\psi + \phi) \ln \frac{\psi + \phi}{1 + \phi} \right) \lambda r n^H$ .

**Proof:** Define  $f(\psi) = \left( \psi - 1 - (\psi + \phi) \ln \frac{\psi + \phi}{1 + \phi} \right)$ . Note that  $f(1) = 0$ .

Since  $f'(\psi) = -\ln \frac{\psi + \phi}{1 + \phi} > 0 \Rightarrow f(\psi) < 0$  for  $0 < \psi < 1$ . So,  $\mu^L < 0$ . □

### Appendix B:

**Lemma B1:** Let  $w_t(\lambda)$  be a Brownian Motion with absorbing boundaries as defined in (14) and (15),  $\mu$  and  $\sigma^2$  as defined in (18) and (19) and with starting point  $w_T \in (\underline{w}, \bar{w})$ . Let  $\tilde{w}_t$  be another process with the same form and parameters as  $w_t$  up to some possibly stochastic time  $\tau > T$  at which time the parameter  $\lambda$  is replaced by  $\lambda'$ , which may also be stochastic. Both  $w_t(\lambda)$  and  $\tilde{w}_t$  yield the same probabilities of hitting the boundaries which are given by (21) and (23).

**Proof:** Define  $b(\tau, w_\tau, \lambda')$  as the joint p.d.f. of  $\tau$ ,  $w_\tau$  and  $\lambda'$  conditional on not hitting either boundary in  $t \leq \tau$ . Define  $P_{w_T(\lambda) \rightarrow \bar{w}}$  as the probability starting from  $w_T$  that process hits the boundary  $\bar{w}$  before  $\underline{w}$ . Since  $w_t(\lambda)$  is a standard Brownian Motion  $P_{w_0(\lambda) \rightarrow \bar{w}}$  is given by

$$P_{w_0(\lambda) \rightarrow \bar{w}} = 1 - \left[ \left( e^{\frac{-2\mu w}{\sigma^2}} - e^{\frac{-2\mu w_0}{\sigma^2}} \right) / \left( e^{\frac{-2\mu w}{\sigma^2}} - e^{\frac{-2\mu \bar{w}}{\sigma^2}} \right) \right] \quad (30)$$

and  $P_{w_i(\lambda) \rightarrow \underline{w}} = 1 - P_{w_i(\lambda) \rightarrow \bar{w}}$ . These depend on  $\lambda$  only through the ratio  $\mu / \sigma^2$ . Hence the probabilities of  $w_t(\lambda)$  hitting the boundaries do not depend on  $\lambda$ . Although the date  $\tau$  as no special relevance to this

process we can still decomposed this probability into the probability that it transitions before or at  $\tau$  and the probability it transitions after  $\tau$  :

$$P_{w_0(\lambda) \rightarrow \bar{w}} = P_{w_0(\lambda) \rightarrow \bar{w} | t \leq \tau} + \left[ 1 - P_{w_0(\lambda) \rightarrow \bar{w} | t \leq \tau} - P_{w_0(\lambda) \rightarrow \underline{w} | t \leq \tau} \right] \int_{\tau} \int_{w_{\tau}} \int_{\lambda'} P_{w_{\tau}(\lambda) \rightarrow \bar{w}} b(\tau, w_{\tau}, \lambda') \cdot d\tau \cdot dw_{\tau} \cdot d\lambda' \quad (31)$$

While we know the left-hand side of this, the formulas for the conditional probabilities and p.d.f.s on the right-hand side are unknown. However, since  $\tilde{w}_t$  starts off as the same process we can similarly decompose its probability as:

$$P_{\tilde{w}_0 \rightarrow \bar{w}} = P_{w_0(\lambda) \rightarrow \bar{w} | t \leq \tau} + \left[ 1 - P_{w_0(\lambda) \rightarrow \bar{w} | t \leq \tau} - P_{w_0(\lambda) \rightarrow \underline{w} | t \leq \tau} \right] \int_{\tau} \int_{w_{\tau}} \int_{\lambda'} P_{w_{\tau}(\lambda') \rightarrow \bar{w}} b(\tau, w_{\tau}, \lambda') \cdot d\tau \cdot dw_{\tau} \cdot d\lambda' \quad (32)$$

Here both the left and right-hand side probabilities are unknown. Nevertheless, since it is the same process up to  $\tau$  these conditional probabilities and p.d.f.s are the same as in (31) with the exception of the continuation probabilities in the integrals. Note however these are simply the probabilities for the Brownian Motion starting from  $w_{\tau}$  with parameter  $\lambda'$  and hence for each potential realization of  $w_{\tau}$  and  $\lambda'$  the probability can be found from (30) by substituting  $w_{\tau}$  for  $w_0$ . As before  $\lambda'$  cancels out from these probabilities. Therefore each  $P_{w_{\tau}(\lambda') \rightarrow \bar{w}}$  in equation (32) is equal to the corresponding  $P_{w_{\tau}(\lambda) \rightarrow \bar{w}}$  in (31) and hence  $P_{w_0(\lambda) \rightarrow \bar{w}} = P_{\tilde{w}_0 \rightarrow \bar{w}}$ . The same argument shows that  $P_{w_0(\lambda) \rightarrow \underline{w}} = P_{\tilde{w}_0 \rightarrow \underline{w}}$ , which completes the proof of the lemma. □

**Proposition B1:** The equilibrium of the game with any finite set  $\Gamma$  of possible contingency trigger points which contains  $\bar{q}$  and  $\underline{q}$ , will yield the same transition probabilities as the baseline model.

**Proof:** From the baseline model where  $\Gamma = \{\underline{q}, \bar{q}\}$  add one contingency trigger point  $q'$ . If  $q' > \bar{q}$  or  $q' < \underline{q}$  then the state  $q'$  would never be reached in the baseline equilibrium and hence we can construct a parallel equilibrium where no agent chooses to have a contingency triggered at  $q'$ . So the addition of  $q'$  will not change the transition probabilities.

If  $q' \in (\underline{q}, \bar{q})$  then some agents may choose to set contingency triggers there. Let  $\pi(q_{\tau})$ , henceforth  $\pi$ , be the probability that an individual agent chooses to set a contingency trigger at  $q'$ . This may be either to buy or to cancel an impending order. The same argument used for  $\lambda$  implies that  $\pi < 1$ . If  $q'$  is a buy

trigger and  $\pi=1$  then each individual would prefer to wait since  $q' < \bar{q}$ . If it is a wait trigger and  $\pi=1$  then each individual would prefer to buy since  $q' > \underline{q}$ .

Moreover if the agents are using  $q'$  as a buy trigger then  $r\pi n^H$  must be finite for all contingencies which may be hit with non-negligible probability. If not then if the contingency is triggered at any  $t < T+1-\varepsilon$  the true value of  $n$ , and hence  $V$ , would be revealed with certainty by the end of the period. Since  $q' < \bar{q}$  each individual would prefer to wait when the state will be revealed with certainty.

So either the addition of  $q'$  has no effect on the outcome in the period ( $\pi=0$ ) or in equilibrium each individual will be indifferent between using it as a trigger or not and the number of agents with outstanding buy orders at each contingency trigger point will be drawn from a Poisson distribution with finite mean. So the addition of the contingency will cause  $\lambda$  to change when it is triggered. But by Lemma C1 this new process has the same transition probabilities as the original process. Iterating the argument allows us to add any finite number of contingency trigger points to the set  $\Gamma$  without altering the transition probabilities.

□

### Appendix C:

**Claim C1:**  $\frac{d\varphi^H}{d\psi} > 0$  where  $\frac{d\varphi^H}{d\psi} = \frac{-2\psi + 2 + (1 + \psi + 2\phi) \ln \frac{\psi + \phi}{1 + \phi}}{(\psi + \phi)(1 + \phi) \left( \ln \frac{\psi + \phi}{1 + \phi} \right)^3}$

**Proof:** Define  $f(\psi) = -2\psi + 2 + (1 + \psi + 2\phi) \ln \frac{\psi + \phi}{1 + \phi}$ . Note that  $f(1) = 0$ .

And  $f'(\psi) = -2 + \frac{1 + \psi + 2\phi}{\psi + \phi} + \ln \frac{\psi + \phi}{1 + \phi}$ . Note that  $f'(1) = 0$ .

$f''(\psi) = \frac{\psi - 1}{(\psi + \phi)^2} < 0$ . Hence  $f'(\psi) > 0$  for  $0 < \psi < 1$ . So  $f(\psi) < 0$ , so  $\frac{d\varphi^H}{d\psi} > 0$ .

□

**Claim C2:**  $\frac{d\varphi^L}{d\psi} > 0$  where  $\frac{d\varphi^L}{d\psi} = \frac{-2\psi + 2 + (1 + \psi + 2\phi) \ln \frac{\psi + \phi}{1 + \phi}}{(\psi + \phi)^2 \left( \ln \frac{\psi + \phi}{1 + \phi} \right)^3}$

**Proof:**  $\frac{d\varphi^L}{d\psi}$  and  $\frac{d\varphi^H}{d\psi}$  have the same numerator. In Lemma A3, it is shown that the numerator is negative.

Since the denominator is negative as well,  $\frac{d\varphi^L}{d\psi} > 0$ .

□

**Claim C3:** The first term of (27) is positive,  $\frac{\partial \text{Prob}(INC)}{\partial w_0} \frac{\partial w_0}{\partial \psi} > 0$ .

**Proof:**  $\frac{\partial \text{Prob}(INC)}{\partial w_0} = -\frac{2\varphi^H e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-\bar{w})}} < 0$ , since  $(1-e^{2\varphi^H(\underline{w}-\bar{w})}) > 0$ . And  $\frac{\partial w_0}{\partial \psi} = -\frac{1}{\psi} < 0$ .  $\square$

**Claim C4:**  $\frac{\partial \text{Prob}(INC)}{\partial \varphi^H} < 0$  where  $\frac{\partial \text{Prob}(INC)}{\partial \varphi^H} = -\frac{1-e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-\bar{w})}} \left[ \frac{2(\underline{w}-\bar{w})e^{2\varphi^H(\underline{w}-\bar{w})}}{1-e^{2\varphi^H(\underline{w}-\bar{w})}} - \frac{2(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{1-e^{2\varphi^H(\underline{w}-w_0)}} \right]$

**Proof:** The term inside the brackets is positive. Examine the properties of the following function:

Let  $g(x) = \frac{2(\underline{w}-x)e^{2\varphi^H(\underline{w}-x)}}{1-e^{2\varphi^H(\underline{w}-x)}}$  where  $x > \underline{w}$ . Then notice that the term inside the brackets is equal to

$[g(\bar{w}) - g(w_0)]$ . If  $\frac{dg(x)}{dx} > 0$ , then  $[g(\bar{w}) - g(w_0)] > 0$  and  $\frac{\partial \text{Prob}(INC)}{\partial \varphi^H} < 0$ .

$$\frac{dg(x)}{dx} = \frac{2e^{2\varphi^H(\underline{w}-x)}}{(1-e^{2\varphi^H(\underline{w}-x)})^2} (e^{2\varphi^H(\underline{w}-x)} - 1 - 2\varphi^H(\underline{w}-x))$$

Let  $z = 2\varphi^H(\underline{w}-x)$  where  $z < 0$  for  $x \in (w_0, \bar{w})$ .

The term in the parenthesis is then equal  $f(z) = e^z - 1 - z$

$f(z=0)=0$ . And  $f'(z) = e^z - 1 < 0$  for all  $z < 0$ . Hence  $\frac{dg(x)}{dx} > 0$ . Therefore  $\frac{\partial \text{Prob}(INC)}{\partial \varphi^H} < 0$ .  $\square$

**Claim C5:** The first term of (29) is negative,  $\frac{\partial \text{Prob}(IPC)}{\partial w_0} \frac{\partial w_0}{\partial \psi} < 0$ .

**Proof:**  $\frac{\partial \text{Prob}(IPC)}{\partial w_0} = \frac{2\varphi^L e^{2\varphi^L(\underline{w}-w_0)}}{1-e^{2\varphi^L(\underline{w}-\bar{w})}} > 0$  since  $\varphi^L < 0$  and  $(1-e^{2\varphi^L(\underline{w}-\bar{w})}) < 0$ . And  $\frac{\partial w_0}{\partial \psi} = -\frac{1}{\psi} < 0$ .  $\square$

**Claim C6:**  $\frac{\partial \text{Prob}(IPC)}{\partial \varphi^L} > 0$  where  $\frac{\partial \text{Prob}(IPC)}{\partial \varphi^L} = \frac{1-e^{2\varphi^L(\underline{w}-w_0)}}{1-e^{2\varphi^L(\underline{w}-\bar{w})}} \left[ \frac{2(\underline{w}-\bar{w})e^{2\varphi^L(\underline{w}-\bar{w})}}{1-e^{2\varphi^L(\underline{w}-\bar{w})}} - \frac{2(\underline{w}-w_0)e^{2\varphi^L(\underline{w}-w_0)}}{1-e^{2\varphi^L(\underline{w}-w_0)}} \right]$

**Proof:** The term inside the brackets is positive. Examine the properties of the following function:



Let  $g(x) = \frac{2(\underline{w}-x)e^{2\phi^L(\underline{w}-x)}}{1-e^{2\phi^L(\underline{w}-x)}}$  where  $x > \underline{w}$  then notice that the term inside the brackets is equal to

$[g(\bar{w}) - g(w_0)]$ . If  $\frac{dg(x)}{dx} > 0$ , then  $\frac{\partial \text{Prob}(IPC)}{\partial \phi^L} > 0$  since  $\bar{w} > w_0$ .

$$\frac{dg(x)}{dx} = \frac{2e^{2\phi^L(\underline{w}-x)}}{(1-e^{2\phi^L(\underline{w}-x)})^2} (e^{2\phi^L(\underline{w}-x)} - 1 - 2\phi^L(\underline{w}-x))$$

Let  $z = 2\phi^L(\underline{w}-x)$  where  $z > 0$  for  $x \in (w_0, \bar{w})$ .

The term in the parenthesis is then equal  $f(z) = e^z - 1 - z$

$f(z=0)=0$ . And  $f'(z) = e^z - 1 > 0$  for all  $z > 0$ . Hence  $\frac{dg(x)}{dx} > 0$ . Therefore  $\frac{\partial \text{Prob}(IPC)}{\partial \phi^L} > 0$

□

### Proof of Proposition 3 (Prior Expected Value):

The prior expected value increases due to  $q^*$  ↑ or  $V^H$  ↑ or  $V^L$  ↑.

First notice that  $(1 - e^{2\phi^H \ln(1-\delta)}) > 0$  since  $\phi^H > 0$  and  $(1 - e^{2\phi^L \ln(1-\delta)}) < 0$  since  $\phi^L < 0$ .

The comparative statics are as follows:

$$\frac{d \text{Prob}(INC)}{dq^*} < 0 \quad \text{and} \quad \frac{d \text{Prob}(IPC)}{dq^*} > 0$$

$$\frac{d \text{Prob}(INC)}{dV^H} < 0 \quad \text{and} \quad \frac{d \text{Prob}(IPC)}{dV^H} > 0$$

$$\frac{d \text{Prob}(INC)}{dV^L} < 0 \quad \text{and} \quad \frac{d \text{Prob}(IPC)}{dV^L} > 0$$

Comparative statics with respect to  $q^*$ :

$$\frac{d \text{Prob}(INC)}{dq^*} = \frac{-2\phi^H e^{2\phi^H [\ln(1-q^*) - \ln q^* + \ln \psi - \ln V^H + \ln(-V^L)]}}{(1-q^*)q^*(1-e^{2\phi^H \ln(1-\delta)})} \quad \text{and} \quad \frac{d \text{Prob}(IPC)}{dq^*} = \frac{2\phi^L e^{2\phi^L [\ln(1-q^*) - \ln q^* + \ln \psi - \ln V^H + \ln(-V^L)]}}{(1-q^*)q^*(1-e^{2\phi^L \ln(1-\delta)})}$$

Comparative statics with respect to  $V^H$ :

$$\frac{d\text{Prob}(INC)}{dV^L} = \frac{2\varphi^H e^{2\varphi^H [\ln(1-q^*) - \ln q^* + \ln \psi - \ln V^H + \ln(-V^L)]}}{(1 - e^{2\varphi^H \ln(1-\delta)}) V^L} \text{ and } \frac{d\text{Prob}(IPC)}{dV^H} = \frac{2\varphi^L e^{2\varphi^L [\ln(1-q^*) - \ln q^* + \ln \psi - \ln V^H + \ln(-V^L)]}}{(1 - e^{2\varphi^L \ln(1-\delta)}) V^H}$$

Comparative statics with respect to  $V^L$ :

$$\frac{d\text{Prob}(INC)}{dV^L} = \frac{2\varphi^H e^{2\varphi^H [\ln(1-q^*) - \ln q^* + \ln \psi - \ln V^H + \ln(-V^L)]}}{(1 - e^{2\varphi^H \ln(1-\delta)}) V^L} \text{ and } \frac{d\text{Prob}(IPC)}{dV^L} = \frac{-2\varphi^L e^{2\varphi^L [\ln(1-q^*) - \ln q^* + \ln \psi - \ln V^H + \ln(-V^L)]}}{(1 - e^{2\varphi^L \ln(1-\delta)}) V^L}$$

□

### Proof of Proposition 5 (Signal Quality):

**Step 1:**  $\frac{d\text{Prob}(INC)}{d\psi} > 0$ , when  $w_0 \rightarrow \underline{w}$ .  $\frac{d\text{Prob}(INC)}{d\psi} < 0$  when  $w_0$  is not to close to either of the bounds,

$$\underline{w} \ll w_0 \ll \bar{w}.$$

**Proof:**

$$i) \frac{d\text{Prob}(INC)}{d\psi} > 0, \text{ when } w_0 \rightarrow \underline{w}.$$

$w_0$  is a function of  $q^*$ . All other terms,  $\underline{w}$ ,  $\bar{w}$ ,  $\mu^H$  and  $(\sigma^2)^H$  in (20) are invariant of  $q^*$ .  $w_0 \rightarrow \underline{w}$  is equivalent to  $\ln \frac{q^*}{1-q^*} \rightarrow \ln(-V^L) - \ln(V^H) + \ln \psi$ . When  $w_0 \rightarrow \underline{w}$  the second term of (27), goes to zero. The first term

remains positive. Hence  $\frac{d\text{Prob}(INC)}{d\psi} > 0$  when  $w_0 \rightarrow \underline{w}$ .

$$ii) \frac{d\text{Prob}(INC)}{d\psi} < 0 \text{ when } w_0 \text{ is not to close to either of the bounds, } \underline{w} \ll w_0 \ll \bar{w}.$$

$$\frac{d\text{Prob}(INC)}{d\psi} = \left( \frac{\partial \text{Prob}(INC)}{\partial w_0} \frac{\partial w_0}{\partial \psi} \right) + \left( \frac{\partial \text{Prob}(INC)}{\partial \varphi^H} \frac{\partial \varphi^H}{\partial \psi} \right)$$

First examine,

$$\frac{\partial \text{Prob}(INC)}{\partial \varphi^H} = - \frac{1 - e^{2\varphi^H(\underline{w}-w_0)}}{1 - e^{2\varphi^H(\underline{w}-\bar{w})}} \left[ \frac{2(\underline{w}-\bar{w})e^{2\varphi^H(\underline{w}-\bar{w})}}{1 - e^{2\varphi^H(\underline{w}-\bar{w})}} - \frac{2(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{1 - e^{2\varphi^H(\underline{w}-w_0)}} \right]$$

Let  $g(x) = \frac{2(\underline{w}-x)e^{2\varphi^H(\underline{w}-x)}}{1 - e^{2\varphi^H(\underline{w}-x)}}$  where  $x > \underline{w}$ . Notice that  $g(x) < 0$  for  $x > \underline{w}$  and  $\frac{dg(x)}{dx} > 0$ .

Pick  $\delta = \delta^*$  such that  $g(\bar{w}) = \frac{1}{2}g(w_0)$ . Then  $[g(\bar{w}) - g(w_0)] = -\frac{1}{2}g(w_0) > 0$ .

For  $\delta^*$ :

$$\frac{\partial \text{Prob}(INC)}{\partial \varphi^H} = \frac{1 - e^{2\varphi^H(\underline{w}-w_0)}}{1 - e^{2\varphi^H(\underline{w}-\bar{w})}} \left[ \frac{(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{1 - e^{2\varphi^H(\underline{w}-w_0)}} \right]$$

From Claim C1 and C3:

$$\frac{d \text{Prob}(INC)}{d\psi} = \left\{ \frac{2\varphi^H e^{2\varphi^H(\underline{w}-w_0)}}{\psi(1 - e^{2\varphi^H(\underline{w}-\bar{w})})} + \left( \frac{\partial \varphi^H}{\partial \psi} \right) \frac{(\underline{w}-w_0)e^{2\varphi^H(\underline{w}-w_0)}}{(1 - e^{2\varphi^H(\underline{w}-\bar{w})})} \right\}$$

Factor out  $\frac{e^{2\varphi^H(\underline{w}-w_0)}}{(1 - e^{2\varphi^H(\underline{w}-\bar{w})})} > 0$ . We then have,

$$\frac{d \text{Prob}(INC)}{d\psi} = \frac{e^{2\varphi^H(\underline{w}-w_0)}}{(1 - e^{2\varphi^H(\underline{w}-\bar{w})})} \left\{ \frac{2\varphi^H}{\psi} + \left( \frac{\partial \varphi^H}{\partial \psi} \right) (\underline{w}-w_0) \right\}$$

The multiplicative term outside the parenthesis is positive. Inside the parentheses  $\varphi^H > 0$  and  $\frac{\partial \varphi^H}{\partial \psi} > 0$  by

Claim C1. By picking  $q^*$  we can have  $w_0$  as big as we like and we can still maintain  $g(\bar{w}) = \frac{1}{2}g(w_0)$  for

a big enough  $\delta$ . Hence  $\frac{d \text{Prob}(INC)}{d\psi} < 0$  for  $\underline{w} \ll w_0 \ll \bar{w}$ .

□

**Step 2:**  $\frac{d \text{Prob}(IPC)}{d\psi} < 0$ , when  $w_0 \rightarrow \bar{w}$ .  $\frac{d \text{Prob}(IPC)}{d\psi} > 0$  when  $w_0$  is not too close to either of the bounds,

$\underline{w} \ll w_0 \ll \bar{w}$ .

**Proof:**

i)  $\frac{d \text{Prob}(IPC)}{d\psi} < 0$ , when  $w_0 \rightarrow \bar{w}$ .  $w_0$  is a function of  $q^*$ . All other terms,  $\underline{w}$ ,  $\bar{w}$ ,  $\mu^H$  and  $(\sigma^2)^H$  in (22)

are invariant of  $q^*$ .  $w_0 \rightarrow \bar{w}$  is equivalent to:

$\ln \frac{q^*}{1 - q^*} \rightarrow \ln(-V^L) - \ln(V^H) - \ln(1 - \delta) + \ln \psi$ . When  $w_0 \rightarrow \bar{w}$  the second term of (29), goes to zero. The

first term of (29) remains negative. Hence  $\frac{d \text{Prob}(IPC)}{d\psi} < 0$ .

ii)  $\frac{d \text{Prob}(IPC)}{d\psi} > 0$  when  $w_0$  is not to close to either of the bounds,  $\underline{w} < w_0 < \bar{w}$ .

$$\frac{d \text{Prob}(IPC)}{d\psi} = \left( \frac{\partial \text{Prob}(IPC)}{\partial w_0} \frac{\partial w_0}{\partial \psi} \right) + \left( \frac{\partial \text{Prob}(IPC)}{\partial \varphi^L} \frac{\partial \varphi^L}{\partial \psi} \right)$$

First examine,

$$\frac{\partial \text{Prob}(IPC)}{\partial \varphi^L} = \frac{1 - e^{2\varphi^L(\underline{w} - w_0)}}{1 - e^{2\varphi^L(\underline{w} - \bar{w})}} \left[ \frac{2(\underline{w} - \bar{w})e^{2\varphi^L(\underline{w} - \bar{w})}}{1 - e^{2\varphi^L(\underline{w} - \bar{w})}} - \frac{2(\underline{w} - w_0)e^{2\varphi^L(\underline{w} - w_0)}}{1 - e^{2\varphi^L(\underline{w} - w_0)}} \right]$$

Let  $g(x) = \frac{2(\underline{w} - x)e^{2\varphi^L(\underline{w} - x)}}{1 - e^{2\varphi^L(\underline{w} - x)}}$  where  $x > \underline{w}$ . notice that  $g(x) > 0$  for  $x > \underline{w}$  and  $\frac{dg(x)}{dx} > 0$ .

Pick  $\delta = \delta^*$  such that  $g(\bar{w}) = \frac{1}{2}g(w_0)$ . Then  $[g(\bar{w}) - g(w_0)] = -\frac{1}{2}g(w_0) < 0$ .

For  $\delta^*$ :

$$\frac{\partial \text{Prob}(IPC)}{\partial \varphi^L} = \frac{1 - e^{2\varphi^L(\underline{w} - w_0)}}{1 - e^{2\varphi^L(\underline{w} - \bar{w})}} \left[ \frac{(\underline{w} - w_0)e^{2\varphi^L(\underline{w} - w_0)}}{1 - e^{2\varphi^L(\underline{w} - w_0)}} \right]$$

From Claim C2 and C4:

$$\frac{d \text{Prob}(IPC)}{d\psi} = \left\{ -\frac{2\varphi^L e^{2\varphi^L(\underline{w} - w_0)}}{\psi(1 - e^{2\varphi^L(\underline{w} - \bar{w})})} + \left( \frac{\delta \varphi^L}{\delta \psi} \right) \frac{(\underline{w} - w_0)e^{2\varphi^L(\underline{w} - w_0)}}{(1 - e^{2\varphi^L(\underline{w} - \bar{w})})} \right\}$$

Let us now factor out  $\frac{e^{2\varphi^L(\underline{w}-w_0)}}{(1-e^{2\varphi^L(\underline{w}-\bar{w})})}$ . We then have:

$$\frac{d \text{Prob}(INC)}{d\psi} = \frac{e^{2\varphi^L(\underline{w}-w_0)}}{(1-e^{2\varphi^L(\underline{w}-\bar{w})})} \left\{ -\frac{2\varphi^L}{\psi} + \left( \frac{\delta\varphi^L}{\delta\psi} \right) (\underline{w} - w_0) \right\}$$

The term outside the parenthesis is negative. Inside the parentheses  $\varphi^L < 0$ , and  $\frac{\delta\varphi^L}{\delta\psi} > 0$ . By picking  $q^*$

we can have  $w_0$  as big as we like and we can still maintain  $g(\bar{w}) = \frac{1}{2}g(w_0)$  for a big enough  $\delta$ . Hence

$$\frac{d \text{Prob}(IPC)}{d\psi} > 0 \text{ for } \underline{w} < w_0 < \bar{w}.$$

□