Aggregation of Information and Beliefs: Asset Pricing Lessons from Prediction Markets*

Marco Ottaviani†  Peter Norman Sørensen‡

July 2012

Abstract

This paper analyzes how asset prices in a binary market react to information when traders have heterogeneous prior beliefs. We show that the competitive equilibrium price under-reacts to information when there is a bound to the amount of money traders are allowed to invest. Under-reaction is more pronounced when prior beliefs are more heterogeneous. Even in the absence of exogenous bounds on the amount traders can invest, prices under-react to information provided traders become less risk averse as their wealth increases. In a dynamic setting, under-reaction results in initial momentum and then over-reaction and reversal in the long run.

Keywords: Aggregation of heterogeneous beliefs, Price reaction to information, Wealth effects.

JEL Classification: D82 (Asymmetric and Private Information), D83 (Search; Learning; Information and Knowledge), D84 (Expectations; Speculations).


†Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208-2013. Phone: +1– 847– 467– 0684. Fax: +1– 847– 467– 1777. E-mail: m-ottaviani@northwestern.edu. Web: http://www.kellogg.northwestern.edu/Faculty/ottaviani/homepage/.

‡Department of Economics, University of Copenhagen, Øster Farimagsgade 5, Building 26, DK–1353 Copenhagen K, Denmark. Phone: +45–3532–3056. Fax: +45–3532–3000. E-mail: peter.sorensen@econ.ku.dk. Web: http://www.econ.ku.dk/sorensen.
1 Introduction

This paper investigates how asset prices relate to the beliefs of traders in financial markets. Our analysis uncovers a novel theoretical mechanism through which prices initially underreact to information under the realistic assumption that traders with heterogeneous prior beliefs are subject to wealth effects. This result provides a simple explanation of pricing patterns that are widely documented in asset prices. Under-reaction takes the form of the favorite-longshot bias in prediction markets. In more general financial markets under-reaction is consistent with post-earning announcement drift and stock price momentum. On the flip side, the same mechanism that leads to under-reaction and momentum in the short run also explains the occurrence of over-reaction and reversal in the long run.

Our analysis is inspired by the operation of prediction markets. Prediction markets are trading mechanisms that target unique events, such as the outcome of a presidential election or the identity of the winner in a sport contest. These markets produce forecasts by aggregating the expectations of traders.\footnote{Partly thanks to their track record as forecasting tools, as documented, for instance, by Forsythe et al. (1992) and Berg et al. (2008), prediction markets are attracting growing interest as mechanisms to collect information and improving decision making in business and public policy contexts. See Hanson (1999), Wolfers and Zitzewitz (2004), and Hahn and Tetlock (2005).} Given the simplicity of the trading environment and the availability of data on reasonably exogenous outcome realizations, prediction markets are ideal laboratories to test asset pricing theories.

We initially formulate a trading model for a binary event, such as the outcome of a presidential election. Risk-neutral traders can take positions in two Arrow-Debreu contingent assets, each paying one dollar if the corresponding outcome occurs. Each trader’s initial endowment is constant with respect to the outcome realization.

Given that traders have limited experience with the underlying events, we allow them to have heterogeneous prior beliefs. These initial opinions are subjective and thus are uncorrelated with the realization of the outcome.\footnote{For the purpose of our analysis, traders’ subjective prior beliefs play the role of exogenous parameters, akin to the role played by preferences.} Having different prior beliefs, traders gain from trading actively. While trade may be motivated by the heterogeneity of prior beliefs, traders also have access to information about the eventual realization of the outcome on which the market is liquidated. Information has an objective nature because it is correlated with the outcome. This conceptual distinction between prior beliefs and information
is standard—as Aumann (1976) notes, “reconciling subjective probabilities makes sense if it is a question of implicitly exchanging information, but not if we are talking about ‘innate’ differences in priors.” To sharpen our result, we mostly assume that all traders agree on their heterogeneous priors and interpret information in the same way, so that beliefs are concordant in Milgrom and Stokey’s (1982) terminology.

Given that traders agree on how to interpret information, differences in the posterior beliefs of traders are uniquely due to differences in prior beliefs. How does the equilibrium price react to information that becomes publicly available to all traders? How does the market price aggregate the traders’ posterior beliefs? Taking into account a typical institutional feature of prediction markets, we first formulate a static market model where risk-neutral traders are restricted to invest a limited amount of money in the market. Our main questions are addressed by a comparative statics analysis of how the market price depends on changes in the public information.

Our first contribution is the observation that the market price does not behave like a posterior belief, but rather systematically under-reacts to information. To understand the mechanism driving under-reaction in the static model, consider a hypothetical market based on which team, Italy or Denmark, will win a soccer game. Suppose that those traders who are subjectively more optimistic about Italy winning live further south. In equilibrium, traders living south of a certain threshold latitude bet all they can on the asset that pays if Italy wins. Likewise, traders north of the threshold latitude invest in the Denmark asset the maximum amount of money allowed.

Now, what happens when traders observe information more in favor of Italy winning? This information causes the price of the Italy asset to be higher, while contemporaneously reducing the price of the Denmark asset, compared to the case with less favorable information. As a result, the southern traders (who are optimistic about Italy) are able to buy fewer Italy assets, which are now more expensive. Similarly, the northern traders can afford, and thus demand, more Denmark assets, now cheaper. Hence, the market would have an excess supply of the Italy asset and excess demand for the Denmark asset. For the market to equilibrate, some northern traders must turn to the Italian side. In summary, when information more favorable to an outcome is available, the marginal trader who determines the price has a prior belief that is less favorable to that outcome. Through this

---

3For example, traders in the Iowa Electronic Markets are allowed to wager up to $500.
countervailing adjustment, the heterogeneity in priors dampens the effect of information on the price.

This under-reaction result amends the common interpretation that the price of an Arrow-Debreu asset represents the belief held by the market about the probability of the event. The reason why the price does not behave like a posterior belief is that here is no unique “market prior” belief for which the equilibrium price is the Bayesian posterior update that incorporates the available information. Instead, the marginal trader’s prior changes in the direction opposite to information. This under-reaction is consistent with the widespread observation of the favorite-longshot bias in betting markets, whereby prices of favorites underestimate the corresponding empirical probabilities, while prices of longshots overestimate them. Our testable prediction that the extent of under-reaction increases in the level of belief heterogeneity seems to be borne out by the data.

The under-reaction result is most striking in a setting with concordant beliefs because then relative asset prices do not capture the traders’ agreement on how to adjust their beliefs. As we also show, under-reaction holds more generally when traders have common priors but interpret information differently and thus have non-concordant beliefs. What is key is that information induces a wealth effect and thus results in a change of the marginal trader.

Our explanation for the favorite-longshot bias is crucially different from the one proposed by Ali (1977) in a pioneering paper and recently revived by Manski (2006), Gjerstad (2005), and Wolfers and Zitzewitz (2005) in the fledgling literature on prediction markets. They analyze the relation between the equilibrium price and the median (or average) belief of traders, possibly depending on the traders’ preferences for risk. The bias is explained with the auxiliary assumption that the median (or average) belief corresponds to the empirical probability. But this assumption is contentious. If the traders’ beliefs really have information content, their positions should depend on the information about these beliefs that is contained in the market price. This tension underlies the modern information economics critique of the Walrasian approach to price formation with heterogeneous beliefs (see the discussion in Chapter 1 of Grossman, 1989). To the prediction markets literature, we contribute the observation that the favorite-longshot bias results without making any assumptions on how the beliefs of the median member of the population relates to the empirical probability.
Our second contribution shows that the under-reaction result does not depend on the trading restrictions of prediction markets. We formulate a more general environment of an asset market with risk averse traders. To encompass the first model as a special case, we conveniently allow—but do not require—traders to be exogenously constrained by their wealth or by the amount they can borrow. As in Milgrom and Stokey (1982), traders can have arbitrary risk preferences, heterogeneous priors, and concordant information. To their well-known characterization of equilibrium, we add a comparative statics analysis of the first-round equilibrium price with respect to information. We show that under-reaction holds under the empirically plausible assumption that traders have decreasing absolute risk aversion, even when no exogenous bound is imposed on the traders’ wealth. The logic is the same as in our baseline model. When favorable information is revealed, traders who take long positions on the asset that now becomes more expensive suffer a negative wealth effect. Hence these traders become more risk averse and cut back their positions.

The identity of the marginal trader and, more generally, the weights accorded to the beliefs of different traders are endogenously determined and depend in a predictable way on the realized information. Thus, we combine elements from the classic “average investor” view (mostly based on asset pricing models with either mean-variance or CARA preferences) with the “marginal investor” view à la Miller (1977). On the one hand, equilibrium is determined by a market clearing condition in line with the average investor view. On the other hand, price reflects the belief of the marginal investor (in the model with wealth constraints) or the quantity-weighted beliefs of inframarginal investors (in the model with risk aversion). Wealth effects imply that the weights assigned to traders with different beliefs change in the opposite direction to the information revealed.

Our analysis integrates results from the classic literature on belief aggregation (Wilson, 1968, Lintner, 1969, and Rubinstein, 1974) with work on the reaction of prices to information (Grossman, 1976). As we show, wealth effects generate under-reaction to information because the price assigns an increased weight to traders with beliefs that are contrary to the realized information. Under-reaction results once we relax simultaneously two of Grossman’s (1976) key assumptions, no wealth effects and common prior. First, wealth effects are essential to obtain the result, because otherwise heterogeneous beliefs can be

---

4 According to Mayshar (1983), this view dates back to John Maynard Keynes, John Burr Williams, and James Tobin.
aggregated, as shown by Wilson (1968). As compared to Varian’s (1989) generalization of Grossman (1976) to heterogeneous priors, we further introduce wealth effects by imposing a limit on the amount that traders can invest (in the baseline prediction market model) or, more generally, by relaxing the assumption of constant absolute risk aversion. Second, heterogeneity in beliefs is also essential to obtain under-reaction and cannot be replaced by heterogeneity in endowments across traders.  

The paper’s third contribution is a characterization of the correlation of price changes over time. In a dynamic extension of the model with new information arriving each period as in Milgrom and Stokey (1982), we derive two novel results:

- The first-round under-reaction is immediately followed by price momentum. Intuitively, the arrival of additional information over time partly undoes the initial under-reaction. This first result is consistent with the observation of momentum—a long-standing puzzle documented by a large empirical literature in finance (for example see Jagadeesh and Titman, 1993, and Bernard and Thomas, 1989).

- The initial under-reaction implies a subsequent over-reaction and reversal, given that the marginal trader has contrarian beliefs. Thus, long term price changes are negatively correlated with medium horizon price changes. Over-reaction and reversals are also consistent with empirical evidence (see DeBondt and Thaler, 1985, Fama and French, 1992, and Lakonishok, Shleifer, and Vishny, 1994).

There is a substantial theoretical literature about price reaction to information. In one strand of the literature, market prices deviate from fundamental values because of patterns in noise trading. For example, in Serrano-Padial (2010) rational traders are unable to correct certain kind of mispricing by naive traders, resulting in overpricing low values and underpricing high values. In a dynamic setting, Cespa and Vives (2009) obtain under-reaction or over-reaction depending on the opaqueness surrounding liquidation value and the predictability of noise traders. Our mechanism delivers realistic pricing patterns without making assumption on the exogenous process governing the dynamic arrival of noise traders.

\footnote{Supplementary Appendix C notes that heterogeneity in endowments permits a classic aggregation result for a class of preferences with wealth effects, Hyperbolic Absolute Risk Aversion with common cautiousness parameter. More favorable information then induces all traders, buyers and sellers, to take more extreme positions. With heterogeneous priors, instead, more favorable information renders buyers less extreme and sellers more extreme.}
Another strand of the literature allows traders to interpret the information incorrectly or differently, thus relaxing concordant beliefs. For example, Barberis, Shleifer, and Vishny (1998) derive momentum by assuming that traders are mistaken about the correct information model, while Hong and Stein (1999) posit that information diffuses gradually and is initially understood only by some traders. Allen, Morris, and Shin (2006) consider overlapping generations of traders who must forecast the next period average forecasts and so end up overweighing the common public information. Banerjee, Kaniel, and Kremer (2009) obtain momentum by assuming that traders do not recognize the information of other traders and thus do not react to the information contained in the equilibrium price. In contrast, we obtain both short-term momentum and long-term reversal even when all traders agree about the correct interpretation of information. Section 4.4 discusses how to obtain more realistic predictions about the volume of trade by relaxing concordant beliefs.

The paper proceeds as follows. Section 2 illustrates our under-reaction result in the context of a prediction market with risk-neutral traders. Section 3 turns to the full model of a financial market with risk-averse traders. Section 4 shows that momentum arises in a natural dynamic extension of the model, and establishes that momentum is naturally followed by reversal. Section 5 concludes. The appendix collects the proofs of the main results. Supplementary Appendix B offers a reinterpretation of our under-reaction result in terms of reaction to the traders’ private information that is revealed in a fully revealing rational expectations equilibrium (REE). Supplementary Appendix C showcases the difference between heterogeneity in prior beliefs and heterogeneity in initial endowments. Supplementary Appendix D reports the relatively standard proof of Proposition 8.

2 Prediction Market Model

Our baseline model is inspired by the rules of the Iowa Electronic Markets for a binary prediction market, in which traders can take positions on whether an event, \( E \), is realized (e.g., the Democratic candidate wins the 2012 presidential election) or not. There are two Arrow-Debreu assets corresponding to the two possible realizations: one asset pays out 1 currency unit if event \( E \) is realized and 0 otherwise, while the other asset pays out 1 currency unit if the complementary event \( E^c \) is realized and 0 otherwise.\(^6\)

\(^6\)The state of the world is given exogenously and cannot be affected by the traders. This assumption is realistic in the case of prediction markets on economic statistics, such as non-farm payroll employment.
Traders enter the market by first obtaining an equal number of both assets. Essentially, the designer of the prediction market initially endows each trader $i$ with $w_{i0}$ units of each of the two assets. One important feature of the market is that there is a limit on how much money each trader can invest. After entering the market, traders can exchange their assets with other traders. A second key feature of the market is that traders are not allowed to hold a negative quantity of either asset. As explained below in more detail, these two restrictions (on the amount of money invested and on the number of assets a trader can sell) impose a bound on the number of asset units that each individual trader can purchase and eventually hold.

Markets clear when the aggregate demand for asset 1 precisely equals the aggregate demand for asset 2. We normalize the sum of the two asset prices to one, and focus on the price $p$ of the asset paying in event $E$.

We assume that there is a continuum $I$ of risk-neutral traders who aim to maximize their subjective expected wealth. Trader $i$ maximizes $\pi_i w_i(E) + (1 - \pi_i) w_i(E^c)$, where $\pi_i$ denotes the trader’s subjective belief. We now turn to the process that determines the trader’s subjective belief, $\pi_i$.

Initially, trader $i$ has subjective prior belief $q_i$. Before trading, all traders can observe a public signal $s$. Conditional on state $\omega \in \{E, E^c\}$, we let $f(s|\omega)$ denote the probability density of the signal. The likelihood ratio for signal realization $s$ is defined as $L(s) = f(s|E)/f(s|E^c)$. The only constraint imposed on the signal distribution is that there is zero probability of fully state-revealing signals, so $L(s) \in (0, \infty)$ with probability one. If trader $i$ observes the realized signal $s$, then by Bayes’ rule the subjective posterior belief $\pi_i$ satisfies

$$\frac{\pi_i}{1 - \pi_i} = \frac{q_i}{1 - q_i} L(s).$$

Hence, $L(s)$ is a sufficient statistic for the signal $s$.

---

7 For example, in the Iowa Electronic Markets each trader cannot invest more than $500. Exemption from anti-gambling legislation is granted to small stake markets created for educational purposes.

8 Our main result (Proposition 2) hinges on the property that this bound is endogenous to the model, because the number of assets each trader eventually holds depends on the market-clearing prices.

9 The results derived in this section immediately extend to the case of risk-loving traders, whose behavior is also to adopt an extreme asset position. We turn to risk-averse traders in Section 3.
For convenience, we normalize the aggregate endowment of each asset to 1. The initial distribution of assets over individuals is described by the cumulative distribution function $G$. Thus $G(q) \in [0, 1]$ denotes the share of all assets initially held by individuals with subjective prior belief less than or equal to $q$. We assume that $G$ is continuous, and that $G$ is strictly increasing on the interval where $G \notin \{0, 1\}$.\footnote{The assumption that the priors are continuously distributed is made to simplify the analysis, but is not essential for our under-reaction result. See also the discussion in Supplementary Appendix B.}

We assume that all traders agree on the conditional distributions $f(s|\omega)$, even though they have heterogeneous prior beliefs—thus posterior beliefs are concordant, the leading case considered by Milgrom and Stokey (1982). Before proceeding, we briefly discuss some of our assumptions:

- Our results crucially depend on the heterogeneity of posterior beliefs across traders. Our departure from the parsimonious assumption that traders share a common prior is motivated mostly on grounds of realism.\footnote{As in most work on heterogeneous priors, prior beliefs are given exogenously in our model. We refer to Brunnermeier and Parker (2005) for a model in which heterogeneous prior beliefs arise endogenously.} In reality, traders are unlikely to have experienced similar events in the past.\footnote{The common prior assumption is sensible when traders are dealing with objective uncertainty and with commonly experienced events, but it is not an implication of rational decision making.} The assumption that traders have concordant beliefs about a publicly observed signal serves to make our main result particularly striking. Even though each and every individual trader’s belief is updated in a Bayes rational way in response to the same information, the equilibrium price moves by less than Bayes’ rule would predict.

- The learning foundations of the notion of Walrasian equilibrium are compatible with the presence of heterogeneous priors. All that is necessary for competitive equilibrium is that traders have access to the price. Learning the strategies of the opponents is not necessary for competitive behavior to result, but is required for the re-interpretation of our results in terms of REE presented in Supplementary Appendix B.

2.1 Competitive Equilibrium

This section characterizes the equilibrium when traders are allowed to exchange assets with other traders in a competitive market. By normalization, the prices of the two assets
sum to one, and we focus on the equilibrium determination of the relative price \( p \) for the asset that pays out in event \( E \).

For every \( L \), trader \( i \)'s demand solves this trader’s maximization problem, given belief \( \pi_i (L) \) satisfying (1), and given market price \( p(L) \). Market clearing requires the price to be such that aggregate net demand is zero, or that the aggregate holding of each asset equals aggregate wealth (normalized to 1).

Solving the choice problem of the risk-neutral traders is straightforward. Suppose trader \( i \) has information with likelihood ratio \( L \) resulting in a posterior belief equal to \( \pi_i \), and suppose that the market price is \( p \). The subjective expected return on the asset that pays out in event \( E \) is \( \pi_i - p \), while the other asset’s expected return is \( (1 - \pi_i) - (1 - p) = p - \pi_i \). With the designer’s constraint on asset portfolios, individual demand thus satisfies the following: if \( \pi_i > p \), trader \( i \) exchanges the entire endowment of the \( E^c \) asset into \( (1 - p) \frac{w_i}{p} \) units of the \( E \) asset. The final portfolio is then \( \frac{w_i}{p} \) units of the \( E \) asset and 0 of the \( E^c \) asset. Conversely, when \( \pi_i < p \), the trader’s final portfolio is 0 of the \( E \) asset and \( \frac{w_i}{(1 - p)} \) of the \( E^c \) asset. Finally, when \( \pi_i = p \), the trader is indifferent between any trade.

**Proposition 1** The competitive equilibrium price, \( p(L) \), is the unique solution to the equation

\[
p = 1 - G \left( \frac{p}{(1 - p) L + p} \right)
\]

and is a strictly increasing function of the information realization \( L \).

**2.2 Under-reaction to Information**

Inverting Bayes’ rule (1), we can always interpret the price as the posterior belief of a hypothetical individual with initial belief \( p(L) / [(1 - p(L)) L + p(L)] \). This implied ex ante belief might be interpreted as an aggregate of the heterogeneous subjective prior beliefs of the individual traders. According to (2), this individual is the marginal trader. However, this aggregation cannot be separated from the realization of information. Our main result states that this belief moves systematically against the information that is initially available to traders.

This systematic change in the market prior against the information implies that the market price under-reacts to information. Consider the inference of any market observer
with a fixed prior belief \( q \). The observer’s posterior probability, \( \pi(L) \), for the event \( E \) satisfies (1), or
\[
\log \left( \frac{\pi(L)}{1 - \pi(L)} \right) = \log \left( \frac{q}{1 - q} \right) + \log L. 
\] (3)
The expression on the left-hand side is the posterior log-likelihood ratio for event \( E \), which clearly moves one-to-one with changes in \( \log L \). Part (ii) of the following Proposition notes that the corresponding expression for the market price, \( \log \left( \frac{p(L)}{(1 - p(L))} \right) \) does not possess this property, but rather moves less than one-for-one with the publicly observable \( \log L \).

Proposition 2  Suppose that beliefs are truly heterogeneous, i.e., the distribution \( G \) is non-degenerate. (i) The marginal trader moves opposite to the information, i.e., the implied ex ante market belief \( p/[ (1 - p) L + p] \) is strictly decreasing in \( L \). (ii) The market price under-reacts to initial information: for any pair \( L' > L \) we have
\[
\log L' - \log L > \log \left( \frac{p(L')}{1 - p(L')} \right) - \log \left( \frac{p(L)}{1 - p(L)} \right) > 0. 
\] (4)
(iii) The market price exhibits a favorite-longshot bias, as there exists a price \( p^* \in [0,1] \) such that \( p(L) > p^* \) implies \( \pi(L) > p(L) \), and \( p(L) < p^* \) implies \( \pi(L) < p(L) \).

To understand the intuition for part (i), consider what happens when traders have information more favorable to event \( E \) (corresponding, say, to the Democratic candidate winning the election), i.e., when \( L \) is higher. According to (2), the price of the \( E \) asset, \( p \), is clearly higher when \( L \) is higher. Now, this means that traders who are optimistic about a Democratic victory can buy fewer units of asset \( E \), because the bound \( w_{10}/p \) is decreasing in \( p \). In addition, traders who are pessimistic about a Democratic victory can buy more units of asset \( E^c \), which they want to buy. If all the traders who were purchasing \( E \) before the increase in \( L \) were still purchasing \( E \) at the higher price that results with higher \( L \), there would be insufficient demand for \( E \). Similarly, there would also be excess demand for \( E^c \). To balance the market it is necessary that some traders who were betting on the Republican candidate before now change sides and put their money on the Democratic candidate. In the new equilibrium, the price must change to move traders from the pessimistic to the optimistic side. Thus the indifferent trader who determines the equilibrium price at the margin holds a more pessimistic prior belief about Democratic victory, the more favorable to Democratic candidate (i.e., the higher) the information, \( L \),
is. Hence, although the price, $p$, rises with the information, $L$, it rises more slowly than a posterior belief, because of this negative effect on the prior belief of the marginal trader.

According to part (iii), under-reaction implies that $\pi (L) > p (L)$ when $p (L)$ is high (so that event $E$ is a favorite) and $\pi (L) < p (L)$ when $p (L)$ is low (longshot). Thus, a favorite-longshot bias results, with longshot outcomes occurring less often than indicated by the price, while the opposite is true for favorites. The favorite-longshot bias is widely documented in the empirical literature on betting markets when comparing winning frequencies with market prices (see Thaler and Ziemba, 1988, Jullien and Salanié, 2008, and Snowberg and Wolfers, 2010).

Re-arranging (4) with (3), we have that $\log (\pi / (1 - \pi)) - \log (p / (1 - p))$ is a strictly increasing function of $p$. Thus, when running the following regression

$$\log \frac{\pi_j}{1 - \pi_j} = a + b \log \frac{p_j}{1 - p_j} + \varepsilon_j,$$

Proposition 2 predicts that $b > 1$. Once we identify the posterior $\pi_j$ chance for an event with the empirical winning frequency corresponding to market price $p_j$, our model thus offers a new informational explanation of the favorite-longshot bias. Outcomes favored by the market occur more often than if the price is interpreted as a probability—and, conversely, longshots win less frequently than the price indicates.

This under-reaction result is driven by the restriction on the amount of money invested (see footnote 7) and, therefore, on the number of assets a trader can sell. In turn, this restriction imposes a bound on the number of assets that each individual can purchase and eventually hold. The result hinges on the fact that this bound (equal to $w_i / p$) is inversely related to the equilibrium price. The result would not hold if the market designer were to impose a direct cap on the number of assets that each trader can buy, rather than on the budget each trader can invest.13

Ali (1977, Theorem 2) provides an antecedent to our under-reaction result. In a model of equilibrium betting with heterogeneous prior beliefs, Ali notes that if the median bettor thinks that one outcome (defined to be the favorite) is more likely than the other, then the equilibrium fraction of parimutuel bets on this favorite outcome is lower than the belief

13Then, over a large range of information realizations, a constant set of optimists (or pessimists) would buy the full allowance of the $E$ (or $E^c$) asset. Since the marginal trader is constant, there would be no under-reaction. However, prediction markets typically bound the traders’ budget, rather than the asset position.
of the median bettor. Under the key assumption that the belief of the median bettor is the correct benchmark for the empirical probability, Ali concludes that favorites are underbet as compared to longshots. However, given that different opinions underlying the heterogeneous priors of traders have no information content by themselves, they should have no bearing on the empirical probabilities, as posited by Ali. In contrast, we make no such assumption and remain agnostic about the relationship between (the distribution of prior) beliefs and the empirical chance of the outcome. Rather, we derive under-reaction as a comparative statics result with respect to the revelation of information.

2.3 Comparative Statics in Prior Beliefs and Wealth

Our equilibrium price \( p(L) \) is determined by (2) which depends on the primitive distribution \( G \) of wealth across traders with different prior beliefs. Changes in this wealth distribution can affect the equilibrium and hence the extent of under-reaction. We show that under-reaction is more pronounced if this distribution is wider. Note that a wider distribution arises in a population where traders simply have greater belief heterogeneity. A wider distribution of wealth over beliefs also arises when more opinionated traders attract more resources, or when less opinionated traders stay away from the market.

In analogy with Rothschild and Stiglitz’s (1970) definition of mean preserving spread, define distribution \( G' \) to be a median-preserving spread of distribution \( G \) if \( G \) and \( G' \) have the same median \( m \) and satisfy \( G'(q) \geq G(q) \) for all \( q \leq m \) and \( G'(q) \leq G(q) \) for all \( q \geq m \).

**Proposition 3** Suppose that \( G' \) is a median-preserving spread of \( G \), denoting the common median by \( m \). Then, more under-reaction results under \( G' \) than under \( G \): \( L > (1 - m) / m \) implies \( \pi(L) > p(L) > p'(L) > 1/2 \), and \( L < (1 - m) / m \) implies \( \pi(L) < p(L) < p'(L) < 1/2 \).

This result is consistent with the observation of more pronounced favorite-longshot bias in political prediction markets, which are naturally characterized by a wider dispersion of beliefs. While cast in a prediction market setting, our baseline model with bounded wealth is also applicable to financial markets where typically traders have a finite wealth and/or can borrow a finite amount of money due to imperfections in the credit market. Empirical
Figure 1: This plot shows the posterior probability for event $E$ as a function of the market price $p$ for the $E$ asset, when the prior beliefs of the risk-neutral traders are uniformly distributed ($\beta = 1$ in the example). The market price is represented by the dotted diagonal.

evidence by Verardo (2009) confirms that momentum profits are significantly larger for portfolios characterized by higher heterogeneity of beliefs.

**Example.** To illustrate our results, suppose that the distribution of subjective prior beliefs over the interval $[0, 1]$ is $G(q) = q^\beta / \left[ q^\beta + (1 - q)^\beta \right]$, where $\beta > 0$ is a parameter that measures the concentration of beliefs. The greater is $\beta$, the less spread is this symmetric belief distribution around the average belief $q = 1/2$. For $\beta = 1$ beliefs are uniformly distributed, as $\beta \to \infty$ beliefs become concentrated near $1/2$, and as $\beta \to 0$ beliefs are maximally dispersed around the extremes of $[0, 1]$. The equilibrium market price $p(L)$ satisfies the linear relation

\[
\log \left( \frac{p(L)}{1 - p(L)} \right) = \frac{\beta}{1 + \beta} \log L.
\]

Hence, $\beta / (1 + \beta) \in (0, 1)$ measures the extent to which the price reacts to information. Price under-reaction is minimal when $\beta$ is very large, corresponding to the case with nearly homogeneous beliefs. Conversely, there is an arbitrarily large degree of under-reaction when beliefs are maximally heterogeneous, corresponding to $\beta$ close to zero.

Assume that a market observer’s prior is $q = 1/2$ for event $E$, consistent with a symmetric market price of $p(1) = 1/2$ in the absence of additional information. The posterior belief associated with price $p$ then satisfies

\[
\log \left( \frac{\pi(L)}{1 - \pi(L)} \right) = \log L = \frac{1 + \beta}{\beta} \log \left( \frac{p(L)}{1 - p(L)} \right).
\]
This provides a particularly strong foundation for the linear regression (5). As illustrated in Figure 1 for the case with uniform beliefs ($\beta = 1$), the market price overstates the winning chance of a longshot and understates the winning chance of a favorite by a factor of two.

2.4 General Belief Heterogeneity

We now investigate how the main under-reaction result extends into a more general model with heterogeneity in the posterior beliefs. After the release of a public signal $L$, traders have heterogeneous posterior beliefs $\pi_i$. Let $G(\pi|L) \in [0, 1]$ denote the share of wealth initially held by individuals with subjective posterior belief less than or equal to $\pi$. Suppose again that $G$ is continuous. In addition, assume that more favorable public information results in higher posterior beliefs in the sense of first-order stochastic dominance: for any pair $L' > L$ and any $\pi$ such that $0 < G(\pi|L)$ or $G(\pi|L') < 1$, we have $G(\pi|L') < G(\pi|L)$. This assumption is weaker than Bayes’ rule.

The heterogeneity of posterior beliefs across traders may stem from a number of sources:

- As a special case, traders may have subjective prior beliefs $q_i$ and concordant information, so that Bayes’ rule as in our baseline model.

- Traders may exhibit over- or underconfidence by assigning too great or too small a weight on their prior belief relative to Bayesian updating, and thus interpret information in a non concordant way, as in Harris and Raviv (1995) and Kandel and Pearson (1996). For example, the posterior belief may satisfy

$$\log \frac{\pi_i (L)}{1 - \pi_i (L)} = \log \frac{q_i}{1 - q_i} + \alpha_i \log L,$$

with $\alpha_i \neq 1$, whereas concordant beliefs require $\alpha_i = 1$. As long as all traders have $\alpha_i > 0$, the ordering assumption on $G(\pi|L)$ follows.

- Traders may have common priors and private signals and trade is subject to noise, as in Diamond and Verrecchia (1981). There is partial information revelation through the publicly observed price (our $L$) in a noisy rational expectations equilibrium, but traders retain some of their initial private information. The belief heterogeneity is
generated by this residual private information. A higher price (higher $L$) is naturally associated with higher posterior beliefs.\footnote{We are not aware of analyses of noisy REE in the present setting with binomial state, risk neutrality and trading constraints. Diamond and Verrecchia (1981) considered a CARA-Normal setting. Their traders’ heterogeneous posterior beliefs are higher in the sense that the posterior variance is constant in $L$, while the posterior means rise with $L$.}

In this more general model, competitive equilibrium results in a price $p$ which solves $p = 1 - G(p|L)$, analogously with (2). Our assumption on $G$ again implies that $p(L)$ is increasing. In analogy with part (i) of Proposition 2, $G(p(L)|L)$ is a strictly decreasing function of $L$. The marginal trader (who has posterior belief equal to the price) is a more pessimistic member of the trader population when the public information, $L$, is more favorable.

**Proposition 4** (i) The competitive equilibrium price, $p(L)$, is the unique solution to the equation $p = 1 - G(p|L)$, and is a strictly increasing function of the information realization $L$. The marginal trader moves opposite to the information, i.e., $G(p(L)|L)$ is strictly decreasing in $L$. (ii) Suppose that equally wealthy individual traders update according to equation (6). Assume that the two individual characteristics $q_i, \alpha_i$ are stochastically independent, that $q_i$ follows a symmetric, unimodal, continuous distribution with median 1/2, and that $\alpha_i$ is distributed continuously and symmetrically around 1 with $\Pr(\alpha_i \in (0,2)) = 1$. Then the market price under-reacts to information, satisfying (4) for any pair $L' > L$.

Part (i) extends part (i) of Proposition 2 to this broader class of posterior belief heterogeneity, without further assumptions. The generalization of the empirically testable part (ii) requires more structure on how the beliefs distribution $G$ varies with information $L$. It is worth remarking that part (ii) of Proposition 4 continues to hold in the limiting case with common prior but heterogeneous confidence parameters $\alpha_i$. When the distribution of $\alpha_i$ is symmetric around 1, the fact that $G(p(L)|L)$ decreases in $L$ implies that the marginal trader has $\alpha_i < 1$ whenever $L \neq 1$.\footnote{Clearly, under-reaction fails once the distribution of $\alpha_i$ assigns sufficiently large weight to large values, for example when most traders over-react to information.}
3 Asset Market Model: Risk Aversion

So far we have assumed that each individual trader is risk neutral, and thus ends up taking as extreme a position as possible on either side of the market. Now, we show that our main result extends nicely to risk-averse individuals, under the empirically plausible assumption that traders’ absolute risk aversion is decreasing with wealth. This result does not rely on imposing exogenous constraints on the wealth traders are allowed to invest.

3.1 Model and Equilibrium

Realistically, suppose that each prediction market trader is initially endowed with the same number $w_0$ of each asset. To properly capture the effect of risk aversion, we suppose that each trader $i$ is also characterized by an initial, state-independent level $W_i$ of additional wealth which cannot be brought into the prediction market.\(^{16}\) Trader $i$ maximizes subjective expected utility of final wealth, $\pi_i u_i (w_i (E)) + (1 - \pi_i) u_i (w_i (E^c))$, where $\pi_i$ is the trader’s subjective belief. We suppose that $u_i$ is twice differentiable with $u_i' > 0$ and $u_i'' < 0$, and satisfies the DARA assumption that the de Finetti-Arrow-Pratt coefficient of absolute risk aversion, $-u_i''/u_i'$, is weakly decreasing with wealth, $w_i$. The cumulative distribution function $H$ now describes the marginal distribution of subjective beliefs, and it is assumed continuous and strictly increasing where $H \notin \{0, 1\}$. Public information $L$ arrives as in the baseline model.

In the market, traders can exchange their asset endowments. To recover the baseline model as a special case (once the coefficient of absolute risk aversion is constant and equal to zero), we allow traders to be subject to a constraint on the wealth they can invest. Note that this constraint does not bind, unless the trader is nearly risk neutral or the difference between $\pi_i$ and $p$ is sufficiently large. We stress that our under-reaction result holds regardless of whether this constraint is binding. In analogy with Proposition 11 we have:

**Proposition 5** There exists a unique competitive equilibrium. The price, $p$, is a strictly increasing function of the information realization $L$.

\(^{16}\) See Musto and Yilmaz (2003) for a model in which, instead, traders are subject to wealth risk, because they are differentially affected by the redistribution associated with different electoral outcomes.
3.2 Belief Aggregation with CARA Preferences

Suppose first that the traders have constant absolute risk aversion (CARA) utility functions, with heterogeneous degrees of risk aversion, such that \( u_i (w) = - \exp (-w/t_i) \), where \( t_i > 0 \) is the constant coefficient of risk tolerance, the inverse of the coefficient of absolute risk aversion. Denoting the relative risk tolerance of trader \( i \) in the population by \( \tau_i = t_i / \int_0^1 t_j dH (q_j) \), we have:

**Proposition 6** Suppose traders have CARA preferences and heterogeneous beliefs. Define an average prior belief \( q \) by

\[
\log \left( \frac{q}{1-q} \right) = \int_0^1 \tau_i \log \left( \frac{q_i}{1-q_i} \right) dH (q_i),
\]

and for each individual let

\[
d^*_i = t_i \log \left( \frac{q_i - qq_i}{q - qq_i} \right). (8)
\]

Suppose that \( 1 + w_0 / \inf_i d^*_i < w_0 / \sup_i d^*_i \). When the realized information \( L \) is in the range satisfying

\[
1 + \frac{w_0}{\inf_i d^*_i} \leq \frac{qL}{qL + 1 - q} \leq \frac{w_0}{\sup_i d^*_i} (9)
\]

then the equilibrium price satisfies Bayes’ rule with market prior \( q \). When \( L \) falls outside this range, the price under-reacts to changes in \( L \).

Risk aversion allows for the possibility that no trader is bound by the trading constraint. This is more likely to happen when \( w_0 \) is large, as also suggested by condition (9). With CARA preferences and when no trader is constrained, trader \( i \) chooses net demand \( d^*_i \) in equilibrium. Under CARA, wealth effects vanish and heterogeneous beliefs can be aggregated, according to formula (7), consistent with the classic result of Wilson (1968), Lintner (1969), and Rubinstein (1974). The market price thus behaves as a posterior belief and there is no under-reaction.

3.3 Under-reaction with DARA Preferences

We have seen that CARA preferences lead to an unbiased price reaction to information when the trading constraints are not binding in equilibrium. Now we verify that, for strict DARA preferences, a bias arises in the price, whether traders are constrained or not. When \( L \) rises, the rising equilibrium price yields a negative wealth effect on any optimistic
individual (with $\pi_i > p$) who is a net demander ($\Delta x_i > 0$). Conversely, pessimistic traders benefit from the price increase. With DARA preferences, the wealth effect implies that optimists become more risk averse while pessimists become less risk averse. Although the price rises with $L$, it is less reactive than a posterior belief, because pessimists trade more heavily in the market when information is more favorable.$^{17}$

**Proposition 7** Suppose that beliefs are truly heterogeneous and that all individuals have strict DARA preferences. Whether wealth constraints are imposed or not, the market price under-reacts to information, satisfying (4) for any pair, $L' > L$.

The asset pricing literature often assumes that traders have a common prior belief (Grossman, 1976). Under the common prior assumption, the price reacts one-for-one to information, regardless of risk attitudes. Our under-reaction result thus hold once we allow for both heterogeneous priors and wealth effects. The intuition for this result is the same as in the baseline model with limited wealth. As $L$ increases, optimists suffer a negative wealth effect, become more risk averse, and thus optimally reduce their demand of the $E$ assets. The converse holds for pessimists. Thus, the equilibrium price adjusts by increasing the weight assigned to traders with prior beliefs less favorable to $E$.

**Example with Logarithmic Preferences.** Suppose traders have logarithmic preferences, $u_i(w) = \log w$, satisfying DARA. In order to highlight the difference between Propositions 2 and 7, namely the inclusion of unconstrained traders, we remove completely the trading constraint. The well-known solution to the individual demand problem with Cobb-Douglas preferences gives $w_i(E) = \pi_i (W_i + w_0) / p$. Let again the cumulative distribution of relative wealth over beliefs be denoted by $G$. The market-clearing price is then a wealth-weighted average of the posterior beliefs,$^{18}$

$$p(L) = \int_0^1 \pi(L) \, dG(q) = \int_0^1 \frac{qL}{qL + (1 - q)} \, dG(q). \quad (10)$$

$^{17}$Given that CARA is the knife-edge case, reversing the logic of Proposition 7 it can be shown that over-reaction results when risk aversion is increasing but not too much (so that demand monotonicity is preserved).

$^{18}$Cobb-Douglas preferences are homothetic, so that wealth expansion paths are linear. With more general utility functions, this property fails, and the extent of under-reaction can be affected by a proportional re-sizing of wealth across the population of traders.
When $G$ is uniform, integration by parts of (10) yields $p(L) = L(L - 1 - \log L) / (L - 1)^2$ for all $L \neq 1$. If $p(1) = \int_0^1 q \, dq = 1/2$ is the prior belief of an outside observer, the favorite-longshot bias can be illustrated in a graph similar to Figure 1.

Edgeworth Box Illustration. We can graphically illustrate the logic of Proposition 7 for a market with two types of traders (with prior beliefs $q_1 < q_2$) and no trading constraints. As shown in Figure 2, the Edgeworth box is a square because there is no aggregate uncertainty. The initial endowment, $e$, lies on the diagonal. Traders have convex indifference curves, which are not drawn to avoid cluttering the picture. The slope of the indifference curves at any safe allocation is $-\pi_i / (1 - \pi_i) = -q_i L / (1 - q_i)$, so along the diagonal trader 2 (optimist) has steeper indifference curves than trader 1 (pessimist). In equilibrium, the marginal rates of substitution are equalized. Thus the equilibrium allocation, $w^*$, lies above the diagonal, where the optimistic buys asset $E$.

How is the equilibrium affected by an exogenous change in information from $L$ to $L' > L$? Marginal rates of substitution are affected such that all indifference curves become
steeper by a factor of $L'/L$. For the sake of argument, imagine that the price were to change as a Bayesian update of market belief $p(L)$ to $p' = p(L) L'/ (p(L) L' + (1 - p(L)) L)$. Since $p' > p(L)$, the new budget line through $e$ passes above $w^*$, illustrating the wealth effect which is positive for the pessimistic trader $1$. Now, as it has been well known since Arrow (1965), DARA implies that the wealth expansion paths diverge from the diagonal. The richer trader $1$ thus demands a riskier bundle further away from the diagonal than at $w^*$, whereas the poorer trader $2$ demands a safer bundle closer to the diagonal. To reach an equilibrium, the price must adjust so as to eliminate the excess demand for asset $E^c$. This is achieved by a relative reduction in the relative price for asset $E$, so that $p(L') < p'$. Thus prices must under-react to information.

4 Dynamic Price Effects

In this section we extend our model to a dynamic setting in which information arrives sequentially to the market after the initial round of trade. We verify that there exists an equilibrium where the initial round of trade is captured by our baseline model, and where there is no trade in subsequent periods, consistent with Milgrom and Stokey’s (1982) no trade theorem. We then show that the initial under-reaction of the price to information implies momentum of the price process in subsequent periods—if the initial price movement is upward, prices subsequently move up on average. Intuitively, first-round information is swamped by the information revealed in subsequent rounds, and hence over time the price comes to approximate the correctly updated prior belief. Under an additional symmetric information assumption, we prove that the initial under-reaction must be followed by subsequent price over-reactions.

4.1 Model

Consider a constant set of traders $I$ who are initially in the same situation as in Section 3.1. Each trader is allowed to trade at every time date $t \in \{1, \ldots, T\}$ at price $p_t$ that is determined competitively. The joint information publicly revealed to traders up until period $t$ has likelihood ratio $L_t$, so that $L_t$ encompasses $L_{t-1}$ and the new information observed in period $t$. The asset position of trader $i$ after trade at period $t$ is summarized by $\Delta x_{it}$. At time $T + 1$ the true event is revealed, and the asset pays out. Each trader aims
to maximize the expected utility of period $T + 1$ wealth, which consists of other wealth $W_i$ plus prediction market wealth $w_{iT}$.

A dynamic competitive equilibrium is defined as follows. First, for every $t = 1, \ldots, T$, there is a price function $p_t(L_t)$. By convention, $p_{T+1} = 1$ when $E$ is true, and $p_{T+1} = 0$ when $E^c$ is true. Second, given these price functions, every trader $i$ chooses a contingent strategy of asset trades in order to maximize expected utility of final wealth. If constrained, the trader’s prediction market wealth must always stay non-negative. Finally, in every period $t$ at any information $L_t$ realization, the market clears.\footnote{The equilibrium of this Proposition is also a fully revealing REE when information is spread across traders, as in Supplementary Appendix B. Namely, we know from before that $p_t(L_t)$ is injective, and it follows from (11) that also $p_t(L_t)$ is injective.}

**Proposition 8** There exists a dynamic competitive equilibrium with the following properties. In the first round of trade, the price $p_1(L_1)$ is the static equilibrium price $p(L_1)$ from Proposition 5. In all subsequent periods there is no trade, and the price satisfies Bayes’ updating rule,

$$\frac{p_t(L_t)}{1 - p_t(L_t)} = \frac{L_t}{L_1} \frac{p_1(L_1)}{1 - p_1(L_1)}.$$  \hspace{1cm} (11)

The marginal trader, who holds belief $p_i(L_1) = p(L_1)$ after the first round of trading, remains the marginal trader in future rounds. The market price in future periods is the Bayesian posterior of this belief updated with the newly arriving information. From this trader’s point of view prices follow a martingale, i.e., $E[p_{t_2}(L_{t_2}) | L_{t_1}] = p_{t_1}(L_{t_1})$ for all $t_2 > t_1 \geq 1$.

Every trader who is initially more optimistic than this marginal trader, and hence has first-round posterior $\pi_i(L_1) > p(L_1)$ and has chosen $\Delta x_i > 0$, believes that the price is a sub-martingale (trending upwards). Despite this belief, the no-trade theorem establishes that such a trader does not wish to alter the position away from the initial $\Delta x_i$. The position already reflects a wealth- or risk-constrained position on the asset eventually rising in price, and there is no desire to further speculate on the upward trend in future asset prices.

### 4.2 Early Under-reaction and Momentum

By Proposition 7, an observer with neutral prior belief $q = p(1)$ sees initial price under-reaction, disagreeing with the marginal trader of belief $\pi_i(L_1) = p(L_1)$. The observer
holds an initial disagreement with the marginal trader of belief $\pi_i (L_1) = p(L_1)$. As more information arrives over time, both the price and the observer’s belief are updated with Bayes’ rule. From the point of view of the observer, how are asset prices expected to develop over time? How does the initial disagreement change over time?

As a benchmark, consider the marginal trader’s expectation. If we let $p_0 (L_0)$ denote this trader’s prior, Bayesian updating implies that prices satisfy the martingale property, $E [p_{t_2} (L_{t_2}) - p_{t_1} (L_{t_1}) | L_{t_1}] = 0$ for all $t_2 > t_1 \geq 0$. The martingale property implies that $E [(p_{t_3} (L_{t_3}) - p_{t_2} (L_{t_2})) (p_{t_2} (L_{t_2}) - p_{t_1} (L_{t_1})) | L_{t_1}] = 0$ for all $t_3 > t_2 > t_1 \geq 0$.20 We show that the outside observer sees a different relation between current and future price changes. After the initial price reaction, prices exhibit momentum, consistent with the empirical findings of Jagadeesh and Titman (1993) and subsequent literature.

**Proposition 9** Suppose that beliefs are truly heterogeneous and that all individuals are constrained or have strictly decreasing absolute risk aversion (DARA). Fix the observer’s prior at the natural level $q = p(1)$. Prices exhibit early momentum, i.e., for any date $t > 1$,

$$E [(p_t (L_t) - p_1 (L_1)) (p_1 (L_1) - p (1)) | L_1] \geq 0,$$

with strict inequality when $L_1 \neq 1$ and the distribution of $L_t / L_1$ is non-degenerate.

Thus, in a regression of early price changes $p_t - p_1$ on initial price reactions $p_1 - q$ there should be a positive coefficient. If we further make the natural assumption for the observer that events $E$ and $E^c$ are symmetric, then $E [p_t (L_t)] = 1/2$ for all $t$. Inequality (12) then implies that early price changes are positively correlated with initial price changes.21

Proposition 9 is also consistent with the seemingly conflicting findings on price drift recently documented by Levitt and Gil (2007) and Croxson and Reade (2008) in the context of sport betting markets. On the one hand, Levitt and Gil (2007) find that the immediate price reaction to goals scored in the 2002 World Cup games is sizeable but incomplete and that price changes tend to be positively correlated, as predicted by our model. On the other hand, Croxson and Reade (2008) find no drift during the half-time

---

20 Under this belief we have $E [(p_{t_3} - p_{t_2}) (p_{t_2} - p_{t_1}) | L_{t_1}] = E [E [p_{t_3} - p_{t_2} | L_{t_2}] (p_{t_2} - p_{t_1}) | L_{t_1}]$ and $E [E [p_{t_3} - p_{t_2} | L_{t_2}]] = 0$ for all $L_{t_2}$.

21 Although the present analysis focuses on the periods that follow an initial period in which trade opens, our results apply more broadly to trading environments in which the arrival of new information coincides with trade—either because of added liquidity reasons or differential interpretation of information, from which the present analysis abstracts away.
break, thus challenging the view that the positive correlation of price changes during play time indicates slow incorporation of information. Consistent with this second bit of evidence, our model predicts the absence of drift when no new information arrives to the market, as it is realistic to assume during the break when the game is not played. These results follow when $L_t = L_{t'} = L_{t''}$ for all periods $t$ in the break $\{t', ..., t''\}$.

### 4.3 Late Over-reaction and Reversal

Momentum suggests a tendency for a correction of the initial price under-reaction over time. As more information arrives and beliefs are updated, the difference in priors plays an ever smaller role. However, there is an additional general effect. Since the belief difference is larger in the beginning, the market tends to catch up more in the beginning than later. From an outside observer’s point of view, the market over-reacts to information after period 1. Assuming symmetric information, we can show that the observer expects an eventual reversal, as $p_t - p_1$ is negatively correlated with the eventual adjustment from $p_t$ to the asset’s fundamental value $p_{T+1} = 1_E$.

**Proposition 10** Suppose that beliefs are truly heterogeneous and that all individuals are constrained or have strictly decreasing absolute risk aversion (DARA). Assume that the information is symmetrically distributed at all dates and fix the observer’s prior at $q = 1/2$. Prices exhibit late reversal, i.e., for any date $t > 1$,

$$E \left[ (p_{T+1} - p_t (L_t)) (p_t (L_t) - p_1 (L_1)) | L_1 \right] \leq 0,$$

with strict inequality provided $L_t$ is not perfectly revealing and the distribution of $L_t/L_1$ is non-degenerate.

This reversal effect is consistent with empirical findings in the asset pricing literature. Note that our heterogeneous beliefs theory predicts that reversal is a necessary counterpart to initial momentum. Both effects are driven by the same initial disagreement between a market observer and the marginal trader.

### 4.4 Discussion

We have been able to fully characterize the time series properties of prices within a model of heterogeneous priors and concordant beliefs thanks to Milgrom and Stokey’s (1982) no-trade-after-the-first-period theorem. The absence of trade greatly simplifies the dynamic
analysis because it rules out the possibility of speculative trade conducted with the scope of reselling in the future to other more optimistic traders, as in Harrison and Kreps (1978).

The literature has long recognized that it is difficult to reconcile the sizeable trading volume that is observed in financial markets with the predictions of traditional asset pricing models. To obtain more realistic predictions for the volume of trade, Harris and Raviv (1993) assume that traders have common priors but heterogeneous confidence parameters \( \alpha_i \), consistent with our specification (6) with non-concordant beliefs. In their model, trade volume is positive only when new information arrives that more than reverses the initial information, \( \log(L_1) \log(L_2) < 0 \). Then, traders who are more optimistic for \( E \) based on \( L_1 \) become more optimistic for \( E^c \) after observing \( L_2 \). Harris and Raviv (1993) further assume that the asset price always equals the value expectation of traders with a constant confidence parameter \( \alpha_i \). Hence, initial under-reaction implies subsequent under-reaction. Modifying Harris and Raviv’s (1993) price formation process, Palfrey and Wang (forthcoming) obtain over-reaction to good news and under-reaction to bad news, but they do not address potential momentum or reversal.

Kandel and Pearson (1995) add the possibility that the arrival of information directly changes the prior belief \( q_i \) of each trader type. With myopic traders, they find that trading volume can be positive even when the price does not change, as empirically observed. Following a similar specification for information, Banerjee and Kremer (2010) derive positive autocorrelation in volume when traders are forward looking in a CARA model.

The combination of Harris and Raviv’s (1993) non-concordant updating with Milgrom and Stokey’s (1982) heterogeneous prior beliefs delivers richer dynamic patterns of prices and trading volume than either model in isolation. To illustrate, consider the posterior belief model (6) with heterogeneity both in prior beliefs \( (q_i) \) as well as in updating confidence factors \( (\alpha_i) \). Start from the equilibrium positions of our static equilibrium. Following any new realization of public information \( L_2 \neq L_1 \), traders are separated into all-\( E \) positions and all-\( E^c \) positions by the iso-posterior \( \pi_i(L_2) = p_2(L_2) \) which represents a new trade-off between the prior \( q_i \) and the confidence parameter \( \alpha_i \). Arrival of new information always induces trade, similar to what is illustrated in Figure 3, as highly confident traders respond

---

\(^{22}\)See also Morris (1996) and the survey by Scheinkman and Xiong (2004).

\(^{23}\)Their model predicts that volume is \( |\beta_0 + \beta_1 \Delta P| \), where \( \beta_0, \beta_1 \) are fixed parameters and \( \Delta P \) is the price change. If \( \beta_0 \neq 0 \), there is positive volume when \( \Delta P = 0 \). However, zero volume results when \( \Delta P = -\beta_0/\beta_1 \), which seems inconsistent with the evidence reported.
more to the information than less confident traders.

We leave a full dynamic analysis of retrading with forward looking traders to future work. Models with dynamic trade open a host of issues. For instance, in the static competitive model, each individual can trade rationally without being concerned with other traders’ beliefs or actions. This is no longer true in a dynamic model with forward-looking traders. Traders have a reason to seek a speculative profit from dynamic trading opportunities, based on their reasoning about future equilibrium prices. This reasoning requires beliefs about the evolution of belief heterogeneity in the trader population.

5 Conclusion

Our analysis takes off from prediction markets. Being particularly simple asset markets in which traders have constant endowments across states of the world, prediction markets offer an ideal setting to investigate how market prices react to information when traders have heterogeneous beliefs. We have shown that information results in a redistribution of wealth across traders with different beliefs. Thus prices tend to under-react to information when traders are subject to wealth effects—either because they are allowed to invest a limited amount or because their absolute risk aversion decreases with wealth. This result is driven by a wealth effect arising because traders with heterogeneous beliefs take speculative net positions. Even in the absence of any exogenous bound on positions or without borrowing constraints, under-reaction holds under the realistic assumption that traders become less risk averse when their wealth increases.

To recap, our contribution builds on the literature on trade with heterogeneous priors and combines it with information updating. In our binary state model, we can allow for wealth effects in a tractable way without making parametric or distributional assumptions. Our model’s three key ingredients (heterogeneous beliefs, information, and wealth effects) are characteristically present not only in prediction markets, but also in more general financial markets (see Hong and Stein 2007). While previous literature has considered the effect of these ingredients either in isolation or in partial combination, the simultaneous presence of all three ingredients delivers both under-reaction and momentum in the short run and over-reaction and reversal in the long run. These results provide a single explanation for an array of empirical findings from financial markets that have typically been explained through separate channels.
We see our analysis as a first step toward understanding price reaction to information in the presence of heterogeneous priors and wealth effects. Future work could also add explicit consideration of borrowing that relaxes differentially the wealth constraints of agents; see Fostel and Geanakoplos (2008) for an initial investigation in this direction. Beyond the setting with two states on which we concentrate in this paper, the wealth effect underlying our results introduces an additional channel through which information affects prices: information about the likelihood of a state relative to a second state can impact the price of the asset for a third state. In turn, the adjustment related to this contagion effect impacts the prices of the assets for the first two states. A general analysis of how prices react to information in the presence of wealth effects and heterogeneous priors is a challenging but promising problem for future research.
Appendix

Proof of Proposition 1. For a given likelihood ratio \( L \), the prior of an individual with posterior belief \( \pi_i \) is, using (1), \( q_i = \pi_i / [(1 - \pi_i) L + \pi_i] \). The \( E^c \) asset is demanded in amount \( w_{i0} / (1 - p) \) by every individual with \( \pi_i < p \), or equivalently \( q_i < p / [(1 - p) L + p] \). The aggregate demand for this asset is then \( G (p / [(1 - p) L + p]) / (1 - p) \). In equilibrium, aggregate demand is equal to aggregate supply, equal to 1, resulting in equation (2).

Next, we establish that the price defined by (2) is a strictly increasing function of \( L \). The left-hand side of (2) is a strictly increasing continuous function of \( p \), which is 0 at \( p = 0 \) and 1 at \( p = 1 \). For any \( L \in (0, \infty) \), the right-hand side is a weakly decreasing continuous function of \( p \), for the cumulative distribution function \( G \) is non-decreasing. The right-hand side is equal to 1 at \( p = 0 \), while it is 0 at \( p = 1 \). Thus there exists a unique solution, such that \( G \notin \{0,1\} \). When \( L \) rises, the left-hand side is unaffected, while the right-hand side rises for any \( p \), strictly so near the solution to (2) by the assumptions on \( G \). Hence, the solution \( p \) must be increasing with \( L \).

Proof of Proposition 2. (i) When \( L \) increases, so does \( p(L) \). By equation (2), when \( p(L) \) increases, \( p(L) / [(1 - p(L)) L + p(L)] \) must fall, because the cumulative distribution function \( G \) is non-decreasing. (ii) By Proposition 1, \( p(L') > p(L) \). Note that (4) is equivalent to

\[
\frac{p(L')}{1 - p(L')} \frac{1}{L'} < \frac{p(L)}{1 - p(L)} \frac{1}{L}.
\]

Using the strictly increasing transformation \( z \rightarrow z / (1 + z) \) on both sides of this inequality, it is equivalent to

\[
\frac{p(L')}{[1 - p(L')] L' + p(L')} < \frac{p(L)}{[1 - p(L)] L + p(L)},
\]

which is true by part (i). (iii) Combining (4) with (3), we see from (ii) that the function

\[
\Psi(L) = \log \left( \frac{\pi(L)}{1 - \pi(L)} \right) - \log \left( \frac{p(L)}{1 - p(L)} \right)
\]

is strictly increasing in \( L \). Hence, one of the following three cases will hold. In the first case, there exists an \( L^* \in (0, \infty) \) such that \( \Psi(L) \) is negative for \( L < L^* \) and positive for \( L > L^* \)—in this case, the result follows with \( p^* = p(L^*) \). In the second case, \( \Psi(L) \) is negative for all \( L \), and the result holds for \( p^* = 1 \). In the third case, \( \Psi(L) \) is positive for all \( L \), and the result is true with \( p^* = 0 \).
Proof of Proposition 3. Note first that \( p((1-m)/m) = 1/2 \) by (2). Consider now \( L > (1-m)/m \) such that the equilibrium prices satisfy \( \pi(L) > p(L), p'(L) > 1/2 \). If, contrary to the claim, \( p(L) < p'(L) \), then (2) implies that 
\[
G \left( \frac{p(L)}{(1-p(L))L + p(L)} \right) = 1 - p(L) > 1 - p'(L) = G' \left( \frac{p'(L)}{(1-p'(L))L + p'(L)} \right).
\]
Further, 
\[
\frac{p'(L)}{(1-p'(L))L + p'(L)} > \frac{p(L)}{(1-p(L))L + p(L)}.
\]
while \( p'(L) > 1/2 \) in equilibrium implies 
\[
\frac{p'(L)}{(1-p'(L))L + p'(L)} < m.
\]
Thus the median preserving spread property implies the contradiction, 
\[
G \left( \frac{p(L)}{(1-p(L))L + p(L)} \right) < G' \left( \frac{p'(L)}{(1-p'(L))L + p'(L)} \right).
\]
A similar argument applies when \( L < (1-m)/m \).

Proof of Proposition 4. (i) At market price \( p \), traders with posterior \( \pi_i > p \) exchange the entire endowment of the \( E^c \) asset into the \( E \) asset, and traders with \( \pi_i < p \) invest all in the \( E^c \) asset. As before, the competitive equilibrium price \( p(L) \) is the unique solution to the market clearing condition \( p = 1 - G(p|L) \). The left hand side of this equation is a continuous, strictly increasing function of \( p \) which is zero at \( p = 0 \) and one at \( p = 1 \), while the right hand side is a continuous, decreasing function of \( p \) which is one at \( p = 0 \) and zero at \( p = 1 \). The left hand side of the equilibrium equation is unaffected by changes in \( L \), while the right-hand side function of \( p \) is strictly higher by the stochastic dominance assumption, so \( p(L) \) is a strictly increasing function of the public likelihood ratio \( L \). Finally, \( G(p(L)|L) = 1 - p(L) \) is a strictly decreasing function of \( L \).

(ii) Figure 3 depicts the space of individual characteristics, with \( \log(q_i/(1-q_i)) \) on the vertical axis. It follows from equation (6) that for any given \( L \), iso-posterior curves (where \( \pi_i(L) \) is constant) are straight lines in this space. In the picture we illustrate the comparative statics effect as the likelihood changes from an initial value \( L \geq 1 \) to a larger value. The less steep line illustrates the iso-posterior for \( \pi_i(L) = p(L) \). If \( L = 1 \), the market clears with \( p(1) = 1/2 \) and the iso-posterior is the horizontal line \( q_i = 1/2 \). Once \( \log(L) > 0 \), iso-posterior lines slope downward. Since \( L > 1 \), we have
Figure 3: Under-reaction with heterogeneity in priors and confidence.

\( p(L) > p(1) = 1/2 \) and \( G(p(L) | L) < 1/2 \). Individuals to the south-west of the iso-posterior have \( \pi_i(L) < p(L) \). Since the characteristics are symmetrically distributed, \( G(p(L) | L) < 1/2 \) implies that the depicted iso-posterior crosses the horizontal axis below the median point \((1,0)\).

When \( L' > L \), iso-posterior lines are steeper. We will first prove that the new equilibrium iso-posterior \( \pi_i(L') = p(L') \) crossed the iso-posterior corresponding to our previous equilibrium at \( \alpha_i < 1 \), as illustrated in Figure 3. Suppose for a contradiction that the intersection has \( \alpha_i \geq 1 \). In that case, the region \( I \) where \( p(L) < \pi_i(L) \) and \( \pi_i(L') < p(L') \) is at least as large as the region \( II \) where \( p(L) > \pi_i(L) \) and \( \pi_i(L') > p(L') \). Using the independence, symmetry and uni-modality assumptions, it is not hard to verify that there is no greater probability of characteristics in region \( II \) than in region \( I \). By implication, \( G(p(L') | L') \geq G(p(L) | L) \) when \( L' > L \). We have thus proved that the intersection has \( \alpha_i < 1 \). By (6), then \( \log (p(L') / (1 - p(L'))) - \log (p(L) / (1 - p(L))) = \alpha_i (\log (L') - \log (L)) < \log (L') - \log (L) \), as claimed.

**Proof of Proposition 5.** Let \( \Delta x_i \) denote the choice variable of trader \( i \), such that \( p \Delta x_i \) units of the \( E^c \) asset are exchanged for \((1 - p) \Delta x_i \) units of the \( E \) asset. Note that this is a zero net value trade, since the asset sale generates \((1 - p) p |\Delta x_i| \) of cash that is spent...
on buying the other asset. The final wealth levels in the two states are:

\[ w_i (E) = W_i + w_0 + (1 - p) \Delta x_i, \]
\[ w_i (E^c) = W_i + w_0 - p \Delta x_i. \]

The trading constraints are that \( \Delta x_i \in [-w_0/(1 - p), w_0/p] \).

The individual trader solves the problem

\[ \max_{\Delta x_i \in [-w_0/(1 - p), w_0/p]} \pi_i u_i (w_i (E)) + (1 - \pi_i) u_i (w_i (E^c)). \]

Strict concavity of \( u_i \) ensures that the maximand \( \Delta x_i \) is unique. By the Theorem of the Maximum, \( \Delta x_i \) is a continuous function of \( \pi_i \) and \( p \). We first show that the optimizer \( \Delta x_i \) is strictly decreasing in \( p \) and weakly increasing in \( \pi_i \), strictly so when \( \Delta x_i \in (-w_0/(1 - p), w_0/p) \).

The constraint set \( [-w_0/(1 - p), w_0/p] \) does not depend on \( \pi_i \) and falls in Veinott’s set order when \( p \) rises. The trader’s objective function \( \pi_i u_i (W_i + w_0 + (1 - p) \Delta x_i) + (1 - \pi_i) u_i (W_i + w_0 - p \Delta x_i) \) has first derivative

\[ \pi_i (1 - p) u_i'(w_i (E)) - (1 - \pi_i) pu_i'(w_i (E^c)) \]

with respect to \( \Delta x_i \). Since \( u_i' > 0 \), the cross-partial of the objective with respect to the choice variable \( \Delta x_i \) and the exogenous \( \pi_i \) is strictly positive, and hence \( \Delta x_i \) is weakly increasing in \( \pi_i \), strictly so when the unique \( \Delta x_i \) optimizer satisfies the interior first-order condition

\[ \frac{\pi_i}{1 - \pi_i} \frac{u_i'(w_i (E))}{u_i'(w_i (E^c))} = \frac{p}{1 - p}. \]

A sufficient condition for a strictly negative cross-partial with respect to \( \Delta x_i \) and \( p \) is

\[ \Delta x_i [\pi_i (1 - p) u_i'(w_i (E)) - (1 - \pi_i) pu_i'(w_i (E^c))] > 0. \]

Using the first-order condition for interior optimality, the second factor of (17) is positive if and only if

\[ -\frac{u_i''(w_i (E^c))}{u_i'(w_i (E^c))} > -\frac{u_i''(w_i (E))}{u_i'(w_i (E))}. \]

By the DARA assumption, this inequality holds if and only if \( w_i (E) > w_i (E^c) \), i.e., \( \Delta x_i > 0 \). Thus the cross-partial is strictly negative, so that \( \Delta x_i \) is strictly decreasing in \( p \).
Equilibrium is characterized by the requirement that the aggregate purchase of asset \( E \) must be zero, i.e., \( \int_0^1 \Delta x_i (p, q_i, L) dH (q_i) = 0 \). When \( p = 0 \), every trader has \( \pi_i > p \) and hence \( \Delta x_i > 0 \), while the opposite relation holds when \( p = 1 \). Individual demands are continuous and strictly decreasing in \( p \), so there exists a unique equilibrium price in \((0,1)\). When \( L \) is increased, \( \pi_i (L) \) rises, and hence \( \Delta x_i \) rises for every trader. The price must then be strictly increased, in order to restore equilibrium.

**Proof of Proposition 6.** Suppose for a moment that no trader is constrained in equilibrium. The necessary and sufficient first-order condition (16) for the unconstrained optimum is solved by

\[
\Delta x_i = t_i \log \left( \frac{1 - p(L)}{p(L)} \frac{\pi_i (L)}{1 - \pi_i (L)} \right). \tag{18}
\]

Market clearing occurs when \( \int_0^1 \Delta x_i dH (q_i) = 0 \). By (18) and using \( \pi_i (L) / (1 - \pi_i (L)) = q_i L / (1 - q_i) \) this is solved by \( p(L) = qL / (qL + 1 - q) \). Inserting this market price in the individual demand (18), the resulting equilibrium demand is \( d_i^* \), as given in (8). This analysis describes the equilibrium, provided no individual is constrained. The lower bound constrains no individual when \( 0 > \inf_i d_i^* \geq -w_0 / (1 - p(L)) \), or equivalently \( p(L) \geq 1 + w_0 / \inf_i d_i^* \). Likewise, the upper bound is equivalent to \( p(L) \leq w_0 / \sup_i d_i^* \).

When a positive mass of traders are constrained, the bias follows from the argument of Proposition 7 reported below.

**Proof of Proposition 7.** The result follows as in the proof of Proposition 2, once we establish that \( \log [p(L) / (1 - p(L))] - \log (L) \) is strictly decreasing in \( L \). Suppose, for a contradiction, that \( \log [p(L) / (1 - p(L))] - \log (L) \) is non-decreasing near some \( L \). First, suppose some traders are constrained by wealth. Traders at the boundary \( \Delta x_i = -w_0 / (1 - p) \) have their demand decreasing in \( p \), and hence \( d\Delta x_i / dL < 0 \). Likewise, \( d\Delta x_i / dL < 0 \) at the other boundary \( \Delta x_i = w_0 / p \). We will argue in the next paragraph that the same effect holds for traders satisfying (16). Since market clearing \( \int_0^1 \Delta x_i (p(L), q_i, L) dH (q_i) = 0 \) implies \( \int_0^1 [d\Delta x_i (p, q_i, L) / dL] dH (q_i) = 0 \), we will then obtain a contradiction establishing the claim.

Second, consider the case with unconstrained traders. Since \( \log[\pi_i (L) / (1 - \pi_i (L))] - \log (L) \) is constant, (16) implies that \( u'(w_i (E) ) / u'(w_i (E^c) ) \) is non-decreasing in \( L \). Using the expressions for the final wealth levels (14) and (15), non-negativity of the derivative
of \( u'_i(w_i(E)) / u'_i(w_i(E^c)) \) implies that
\[
u''_i(w_i(E)) u'_i(w_i(E^c))[(1 - p) \frac{d\Delta x_i}{dL} - \Delta x_i \frac{dp}{dL}] \geq -u''_i(w_i(E^c)) u'_i(w_i(E)) [p \frac{d\Delta x_i}{dL} + \Delta x_i \frac{dp}{dL}] .
\]

The second derivative of the utility function is negative, so this implies
\[
\frac{d\Delta x_i}{dL} \leq \frac{u''_i(w_i(E)) u'_i(w_i(E^c)) - u''_i(w_i(E^c)) u'_i(w_i(E))}{\Delta x_i (1 - p) u'_i(w_i(E)) u'_i(w_i(E^c)) + pu''_i(w_i(E^c)) u'_i(w_i(E))} .
\]

On the right-hand side of (19), \( dp/dL > 0 \) by Proposition 5, and the denominator is negative. Recall that \( \Delta x_i > 0 \) if and only if \( w_i(E) > w_i(E^c) \). By DARA, this implies that
\[
\frac{-u''_i(w_i(E))}{u'_i(w_i(E))} < \frac{-u''_i(w_i(E^c))}{u'_i(w_i(E^c))}
\]
or that the numerator is positive. Likewise, when \( \Delta x_i < 0 \), the numerator is negative. In either case, the right-hand side of (19) is strictly negative. Hence, \( d\Delta x_i/dL \leq 0 \) for every trader who satisfies the first-order condition (16).

**Proof of Proposition 8.** See Supplementary Appendix D for this rather long but straightforward proof.

**Proof of Proposition 9.** The price at \( t \) satisfies Bayes’ rule,
\[
p_t(L_t) = \frac{p_1(L_1) L_t}{p_1(L_1) L_t + (1 - p_1(L_1)) L_1} = p_1(L_1) + \frac{(1 - p_1(L_1)) p_1(L_1) (L_t - L_1)}{p_1(L_1) L_t + (1 - p_1(L_1)) L_1}.
\]
The observer’s posterior at time 1 is \( \pi(L_1) = qL_1 / (qL_1 + 1 - q) \). For the observer,
\[
E[p_t(L_t) - p_1(L_1) | L_1]
= \pi(L_1) E[p_t(L_t) - p_1(L_1) | E, L_1] + (1 - \pi(L_1)) E[p_t(L_t) - p_1(L_1) | E^c, L_1]
= (\pi(L_1) - p_1(L_1)) \{E[p_t(L_t) - p_1(L_1) | E, L_1] - E[p_t(L_t) - p_1(L_1) | E^c, L_1]\},
\]
using the martingale property of Bayes updated prices.

At time 1, there is uncertainty about the realization of the future \( L_t \). Bayes’ rule implies \( L_t/L_1 = f(L_t | E, L_1)/f(L_t | E^c, L_1) \) where \( f \) denotes the p.d.f. for \( L_t \). For any realization of \( L_1 \), we write \( p_t \) for the known \( p_1(L_1) \). Then
\[
E [(p_t(L_t) - p_1(L_1))(p_1(L_1) - q) | L_1]
= (p_1 - q) (\pi(L_1) - p_1) \int_0^\infty (p_t(L_t) - p_1)(L_t - L_1) f(L_t | E^c, L_1) dL_t
= (p_1 - p_1(1)) (\pi(L_1) - p_1) \int_0^\infty \frac{(1 - p_1) p_t^2 (L_t - L_1)}{p_1 L_t + (1 - p_1)L} f(L_t | E, L_1) dL_t.
\]
All terms inside the integral are positive, and the entire integral is positive when \( L_t/L_1 \) has a non-degenerate distribution. By under-reaction, \((p_1(L_1) - p_1(1)) (\pi(L_1) - p_1(L_1)) > 0\) for all \( L_1 \neq 1 \). Hence, \( E[(p_t(L_t) - p_1(L_1)) (p_1(L_1) - q) | L_1] > 0 \) for all \( L_1 \neq 1 \).

**Proof of Proposition 10.** Symmetry states that the distribution of \( L_t/(L_t + L_1) \) conditional on \( (E, L_1) \) is identical to the distribution of \( L_1/(L_1 + L_t) \) conditional on \( (E^c, L_1) \). It follows that \( f(L_t^2/L_t | E, L_1) = (L_t/L_1)^2 f(L_t | E^c, L_1) = (L_t/L_1) f(L_t | E, L_1) \).\(^{24}\) Recall from the text that the expectation of \((p_{T+1} - p_t(L_t)) (p_t(L_t) - p_1(L_1))\) is zero under the marginal trader’s belief implied by \( L_1 \). For the observer, recalling expressions from the proof of Proposition 10, then

\[
E[(p_{T+1} - p_t(L_t))(p_t(L_t) - p_1(L_1)) | L_1] = \pi(L_1) \int_0^\infty (1 - p_t(L_t))(p_t(L_t) - p_1(L_1)) f(L_t | E, L_1) dL_t
\]

\[
- (1 - \pi(L_1)) \int_0^\infty p_t(L_t)(p_t(L_t) - p_1(L_1)) f(L_t | E^c, L_1) dL_t
\]

\[
= (\pi(L_1) - p_1(L_1)) \int_0^\infty (1 - p_t(L_t))(p_t(L_t) - p_1(L_1)) f(L_t | E, L_1) dL_t
\]

\[
+ (\pi(L_1) - p_1(L_1)) \int_0^\infty p_t(L_t)(p_t(L_t) - p_1(L_1)) f(L_t | E^c, L_1) dL_t
\]

\[
= (\pi(L_1) - p_1(L_1)) \int_0^\infty \frac{p_t(L_t)}{p_1(L_1)} (p_t(L_t) - p_1(L_1)) f(L_t | E^c, L_1) dL_t,
\]

where we employed Bayes’ rule

\[
\frac{p_t(L_t)}{1 - p_t(L_t)} = \frac{p_1(L_1)}{1 - p_1(L_1)} \frac{L_t}{L_1} = \frac{p_1(L_1)}{1 - p_1(L_1)} \int f(L_t | E, L_1).
\]

Recalling the expressions for \( p_t(L_t) \) and \( p_t(L_t) - p_1(L_1) \),

\[
\int_0^\infty \frac{p_t(L_t)}{p_1} (p_t(L_t) - p_1) f(L_t | E^c, L_1) dL_t = \int_0^\infty \frac{L_t (1 - p_1)}{[p_1 L_t + (1 - p_1)]^2} f(L_t | E, L_1) dL_t.
\]

We will employ the symmetry property to prove that this integral has the same sign as \( 1 - 2p_1(L_1) \). Thus, over the range from 0 to \( L_1 \), change variables to \( L = L_t^2/L_1 \). The

\(^{24}\)Symmetry implies that \( \Pr(L_t^2/L_1 \leq L_t | E, L_1) = \Pr(L_1/(L_1 + L_t) \leq L_t/(L_t + L_1) | E, L_1) = \Pr(L_1/(L_1 + L_t) \leq L_t/(L_t + L_1) | E^c, L_1) = \Pr(L_t \leq L | E^c, L_1) \). The expression for the conditional densities follows from differentiation with respect to \( L \) on both sides.
integral over this range becomes
\[
\int_{L_1}^{\infty} \frac{L_1 (1 - p_1) p_1 ((L_t^2/|L_t|) - L_1)}{[p_1 (L_t^2/|L_t| + (1 - p_1) L_1)]^2} f \left( \frac{(L_t^2/|L_t|) - (L_t/|L_t|)}{|L_t|} \right) \left( \frac{L_t}{L} \right)^2 dL
\]

\[
= \int_{L_1}^{\infty} \frac{L_1 (1 - p_1) p_1 (L_t - L_1)}{[p_1 L_t + (1 - p_1) L_1]^2} f (L_t|E, L_1) dL_t.
\]

Thus,
\[
\int_{0}^{\infty} \frac{L_1 (1 - p_1) p_1 (L_t - L_1)}{[p_1 L_t + (1 - p_1) L_1]^2} f (L_t|E, L_1) dL_t
\]

\[
= L_1 (1 - p_1) p_1 \int_{L_1}^{\infty} \left( \frac{L_t - L_1}{[p_1 L_t + (1 - p_1) L_1]^2} - \frac{L_t - L_1}{[p_1 L_1 + (1 - p_1) L_1]^2} \right) f (L_t|E, L_1) dL_t.
\]

Observe that
\[
\frac{1}{[p_1 L_t + (1 - p_1) L_1]^2} > \frac{1}{[p_1 L_1 + (1 - p_1) L_1]^2}
\]

if and only if \((2p_1 - 1) (L_t - L_1) < 0\). This holds over the entire range where \(L_t > L_1\)
if and only if \(p_1 < 1/2\). Hence, the entire integral has the same sign as \(1 - 2p_1 (L_1)\), as desired. Its product with \(\pi (L_1) - p_1 (L_1)\) is negative for all \(L_1 \neq 1\): when \(L_1 < 1\) we have \(\pi (L_1) < p_1 (L_1) < 1/2\), while when \(L_1 > 1\) we have \(\pi (L_1) > p_1 (L_1) > 1/2\). In conclusion, 
\[
E \left[ (p_{T+1} - p_t (L_t)) (p_t (L_t) - p_1 (L_1)) \right] < 0.
\]
References


Supplementary Appendix B
Rational Expectations Equilibrium Reinterpretation with Private Information

While in our baseline model information $L$ is publicly revealed, consider now the case in which this information is initially dispersed among market participants, as it is natural in prediction markets. When information is private rather than public, the same outcome results in the unique fully revealing rational expectations equilibrium (REE) à la Grossman (1976) and Radner (1979), as we show below. For this reinterpretation, suppose that $s = \{s_i\}_{i \in I}$, where signal $s_i$ is privately observed by trader $i$ before trading.

The model (i.e., preferences, prior beliefs, and signal distributions) and the rationality of all traders are common knowledge. Before presenting the result, we note that the informational requirements for REE equilibrium are more demanding than those for the notion of competitive equilibrium we have used in the paper. As also argued by Morris (1995), the REE concept in general relies on strong common knowledge assumptions. The learning that is necessary for strategic equilibrium play to become sensible would also eliminate heterogeneity in prior beliefs; see Dekel, Fudenberg, and Levine (2004). In spite of this caveat, REE equilibrium is a useful benchmark, and we find it plausible that individuals make some inferences about the state from the realized price in the presence of asymmetric information. Even though these inferences need not be as correct in reality as they are assumed to be in a REE, it is a strength of our theory that it works also under this narrow, standard assumption.

To solve for a fully revealing REE, suppose that traders behave as if the market price $p$ reveals all information. Traders then form their individual demand functions as before, and market clearing implies that the price functional must be the solution to (2). Given that the solution to (2) is a strictly increasing function of the sufficient statistic $L$ by Proposition 1, the realization of $p$ fully reveals $L$. Thus (2) characterizes the unique fully revealing REE.

**Proposition 11** In the extension of the model with private information, the price $p$ that solves equation (2) is a strictly increasing function of the information realization $L$, so that it characterizes the unique fully revealing REE.

---

25We focus on the necessary properties of this equilibrium, while Radner (1979) discusses sufficient conditions for its existence.
The assumption that prior beliefs are continuously distributed is not essential for the result. If the population is finite, or there are gaps in the distribution, then there can be ranges of information, \( L \), over which the equilibrium price is constant. In that case, the rational expectations equilibrium cannot fully reveal \( L \), but can still reveal the information needed for the proper allocation of the assets. Under-reaction still occurs with respect to the revealed information.

As discussed above, because the assumption that the market reaches a fully revealing equilibrium is not warranted for some market rules, ours is the most optimistic scenario for information aggregation.\(^{26}\) Relaxing the REE assumption, Ottaviani and Sørensen (2009) and (2010) analyze a game-theoretic model of parimutuel betting where traders have a common prior but are unable to condition their behavior on the information that is contained in the equilibrium price.\(^{27}\) They find that the price under-reacts or over-reacts to information depending on the amount of information relative to noise that is present in the market. In contrast, in this paper traders are allowed to perfectly share information, but the price under-reacts to information nevertheless when the priors are heterogeneous and wealth effects are present.

\(^{26}\)Plott and Sunder (1982) and Forsythe and Lundholm (1990) experimentally investigate the conditions leading to equilibrium in settings with differential private information.

\(^{27}\)See also Shin (1991) and (1992) for a derivation of the favorite-longshot bias when prices are quoted by an uninformed monopolist bookmaker. We refer to Ottaviani and Sørensen (2010) for a discussion of (and references to) explanations of the favorite-longshot bias that are not based on information.
Supplementary Appendix C

Heterogeneous Priors versus Heterogeneous Endowments

In Proposition 7, we have obtained under-reaction when motives for trade are generated by the heterogeneity of prior beliefs. It is natural to wonder whether the heterogeneity of priors is an essential ingredient of the result. In particular, would under-reaction occur also when traders are motivated by traditional liquidity motives? In this section, we suppose that traders share a common prior belief but have state-dependent endowments. We find that the price reacts one-for-one to information for a broad class of preferences which includes the logarithmic example from Section 3.3.

We modify the model from that section by assuming that all traders share the common prior \( q \), but allowing trader \( i \)'s initial asset endowment to vary across states, \( w_{i0}(E) \neq w_{i0}(E^c) \). We allow also the aggregate endowment, \( w_0(E) = \int_{i \in I} w_{i0}(E) di \) and \( w_0(E^c) = \int_{i \in I} w_{i0}(E^c) di \), to vary across states. For simplicity, we do not constrain the positions traders can take. Suppose that there exists constants \( \alpha_i \) and \( \beta \) such that trader \( i \) has Hyperbolic Absolute Risk Aversion (HARA), \(-u''_i(w)/u'_i(w) = 1/(\alpha_i + \beta w)\). The fact that \( \beta \) is constant across traders means that traders are equally cautious.\(^{28}\)

**Proposition 12** If all traders have HARA preferences with common cautiousness parameter, then the market price reacts as a Bayesian posterior belief to information.

This assumption guarantees that in this case there exists a representative trader; see also Gorman (1953) and Rubinstein (1974). In equilibrium, this trader must demand the constant aggregate endowment \((w_{i0}(E), w_{i0}(E^c))\). Denoting the utility function of the representative trader by \( U \), the equilibrium price \( p(L) \) must satisfy the equivalent of (16),

\[
\frac{qL}{1 - q} \frac{U'(w_0(E))}{U'(w_0(E^c))} = \frac{p(L)}{1 - p(L)}.
\]

Hence, \( \log[p(L)/(1 - p(L))]-\log(L) \) is constant in \( L \). In the special case without aggregate uncertainty \((w_0(E) = w_0(E^c))\), the market price is equal to the common posterior.

We can briefly revisit the example with logarithmic preferences which belong to the HARA class with \( \alpha_i = 0 \) and \( \beta = 1 \). Denoting by \( \pi \) the common posterior belief, the demand of trader \( i \) satisfies \( pw_i(E) = \pi(pw_{i0}(E) + (1 - p)w_{i0}(E^c)) \). Market clearing implies \( pw_0(E) = \pi(pw_0(E) + (1 - p)w_0(E^c)) \) so that the equilibrium price is \( p = \)

---

\(^{28}\)In the special case with \( \beta > 0 \), the absolute risk aversion is decreasing in \( w \). In this case, these preferences are a special case of DARA preferences. CARA results when \( \beta = 0 \).
This is in accordance with Bayes’ rule, except that the market belief \( p \) is derived from a risk-adjusted prior belief. Trader \( i \)'s net asset trade can be measured by

\[
 w_i (E) - w_{i0} (E) = (1 - \pi) \left( \frac{w_0 (E)}{w_0 (E^c)} w_{i0} (E^c) - w_{i0} (E) \right). 
\]

In this risk-sharing model, trade moves individual asset positions closer to the average. The size of the net trade, \( |w_i (E) - w_{i0} (E)| \), is monotone in \( \pi \). In this setting, when the price responds to information as a posterior belief, buyers buy more aggressively when sellers sell more aggressively. Thus there is no reweighting across traders and no countervailing adjustment in the price depending on the realized information. With heterogeneous beliefs, instead, we found that optimists (who buy) buy less, while pessimists (who sell) sell more when information favors the assets’ underlying event—so the price had to equilibrate against the direction of the information.\(^{29}\)

For more general preferences outside the HARA class the price needs not react as a posterior belief, but it is not immediate whether under or over-reaction results. To understand why the logic of Section 3 does not carry over, reconsider its Edgeworth box illustration. The box is no longer a square when \( w_0 (E) \neq w_0 (E^c) \). Without loss of generality, suppose \( w_0 (E) > w_0 (E^c) \). Unlike in Figure 2, the two risk-averse traders with common prior have an equilibrium bundle on the same side of their respective diagonal (i.e., \( w_1 (E) > w_1 (E^c) \) and \( w_2 (E) > w_2 (E^c) \)). The DARA income expansion paths no longer contradict the possibility of staying in equilibrium when the price is Bayes updated to information. In conclusion, heterogeneity in priors cannot be replaced by heterogeneity in endowments to obtain under-reaction to information.

\(^{29}\)On the optimal allocation of risk with heterogeneous prior beliefs and risk preferences but without private information, see also the recent developments by Gollier (2007).
Supplementary Appendix D
Proof of Proposition 8

We verify that the described outcome is an equilibrium. For the final equilibrium condition, note that the market clears because trader positions are the same as in the static equilibrium. The remainder of the proof verifies that this constant position is indeed optimal in the individual dynamic optimization problem.

Let $\Delta x_{it} (L_t)$ denote the contingent net position of trader $i$ in period $t$ after information realization $L_t$. By convention, $\Delta x_{i0} = 0$. The trader’s wealth evolves randomly over time as $w_{it} (L_t) = w_{it-1} (L_{t-1}) + (p_t (L_t) - p_{t-1} (L_{t-1})) \Delta x_{it-1} (L_{t-1})$ for $t = 1, \ldots, T + 1$, with $w_{i0} > 0$ given as before. If constrained, the trader’s net position choice at $t - 1$ must satisfy $\Delta x_{it-1} (L_{t-1}) = \left[-w_{it-1} (L_{t-1}) / (1 - p_{t-1} (L_{t-1})), w_{it-1} (L_{t-1}) / p_{t-1} (L_{t-1})\right]$.

Suppose at period $t$, information $L_t$ has been realized. To save notation, write $p_t$ for the realization of $p_t (L_t)$ and $w_{it}$ for $w_{it} (L_t)$. Two observations are essential. First, $\Delta x_{it}$ is at the upper bound (interior, lower bound) of the constraint set $[-w_{it} / (1 - p_t), w_{it} / p_t]$ if and only if, for all $L_{t+1}$, $\Delta x_{it+1}$ is on the upper bound (interior, lower bound) of constraint set $[-w_{it+1} (L_{t+1}) / (1 - p_{it+1} (L_{t+1})), w_{it+1} (L_{t+1}) / p_{it+1} (L_{t+1})]$. Second, for all realizations of the string $(L_{t+1}, \ldots, L_T)$, the feasible choice $\Delta x_{iT} (L_T) = \ldots = \Delta x_{it+1} (L_{t+1}) = \Delta x_{it}$ implies

$$\frac{u_i'(W_i + w_i (E))}{u_i'(W_i + w_i (E^c))} = \frac{u_i'(W_i + w_{it} + (1 - p_t) \Delta x_{it})}{u_i'(W_i + w_{it} - p_t \Delta x_{it}).}$$

Both observations follow from the wealth evolution equation $w_{it+1} (L_{t+1}) = w_{it} (L_{t+1}) + (p_{t+1} (L_{t+1}) - p_{t+1} (L_{t+1})) \Delta x_{it} (L_{t+1})$ for periods $\tau = t + 1, \ldots, T$.

To prove our claim that the trader in every period selects the same position $\Delta x_{it} = \Delta x_{i1} (L_1)$ as in the static model given price $p_t (L_t)$, we proceed by backwards induction. The induction hypothesis $t$ states that the agent in period $t$ given price $p_t (L_t)$ (i) chooses $\Delta x_{it}$ to satisfy the static first-order condition

$$\frac{p_t (L_t)}{1 - p_t (L_t)} = \frac{\pi_i (L_t)}{1 - \pi_i (L_t)} \frac{u_i'(W_i + w_{it} (L_t) + (1 - p_t (L_t)) \Delta x_{it})}{u_i'(W_i + w_{it} (L_t) - p_t (L_t) \Delta x_{it})}$$

if feasible, or (ii) chooses $\Delta x_{it} = w_{it} (L_t) / p_t (L_t)$ if the left-hand side of this static condition is below the right-hand side at this choice, and (iii) chooses $\Delta x_{it} = -w_{it} (L_t) / (1 - p_t (L_t))$ if the left-hand side of this static condition exceeds the right-hand side at this choice.

Note from the previous two essential observations, that once we have proved the induction
hypothesis for all $t$, we have $\Delta x_{iT} (L_T) = \ldots = \Delta x_{i1} (L_1)$, and $\Delta x_{i1} (L_1)$ is the solution to the individual problem in Proposition 5.

The induction hypothesis $T$ is satisfied because the static first-order condition characterizes the solution to the remaining one-period problem. We now assume that the induction hypothesis is true at $t + 1, \ldots, T$, and will prove that induction hypothesis $t < T$ is true. Suppose at period $t$, information $L_t$ is realized. Final wealth levels are then

$$w_{iT} (E) = W_i + w_{it} + (p_{t+1} (L_{t+1}) - p_t) \Delta x_{it} + (1 - p_{t+1} (L_{t+1})) \Delta x_{it+1} (L_{t+1})$$

and

$$w_{iT} (E^c) = W_i + w_{it} + (p_{t+1} (L_{t+1}) - p_t) \Delta x_{it} - p_{t+1} (L_{t+1}) \Delta x_{it+1} (L_{t+1})$$

where $\Delta x_{it+1} (L_{t+1})$ is the reaction prescribed by induction hypothesis $t + 1$. The time $t$ problem is

$$\max_{\Delta x_{it} \in [-w_{it}/(1-p_t), w_{it}/p_t]} \pi_i (L_t) E \left[ u_i (w_{iT} (E)) \mid E \right] + (1 - \pi_i (L_t)) E \left[ u_i (w_{iT} (E^c)) \mid E^c \right]$$

where the expectations are taken over the realization of $L_{t+1}$. In case (i), the static first-order condition can be satisfied with an interior choice of $\Delta x_{it}$. Evaluated at this choice, the derivative of the time $t$ objective function is, by the envelope theorem,

$$\pi_i (L_t) E \left[ (p_{t+1} (L_{t+1}) - p_t) u'_i (w_{iT} (E)) \mid E \right] + (1 - \pi_i (L_t)) E \left[ (p_{t+1} (L_{t+1}) - p_t) u'_i (w_{iT} (E^c)) \mid E^c \right] = p_t E \left[ \frac{\pi_i (L_t) u'_i (w_{iT} (E))}{p_t} \left( p_{t+1} (L_{t+1}) - p_t \right) \mid E \right] + (1 - p_t) E \left[ \frac{(1 - \pi_i (L_t)) u'_i (w_{iT} (E^c))}{1 - p_t} \left( p_{t+1} (L_{t+1}) - p_t \right) \mid E^c \right].$$

Here $w_{iT} (E)$ and $w_{iT} (E^c)$ are constant across realizations of $L_{t+1}$. The static first-order condition then allows us to rewrite the derivative with respect to the control variable as

$$\frac{\pi_i (L_t) u'_i (w_{iT} (E))}{p_t} \{ p_t E [p_{t+1} (L_{t+1}) - p_t \mid E] + (1 - p_t) E [(p_{t+1} (L_{t+1}) - p_t) \mid E^c] \}.$$

By the martingale property of Bayes-updated prices at market belief $p_t$, we have

$$p_t E [p_{t+1} (L_{t+1}) - p_t \mid E] + (1 - p_t) E [(p_{t+1} (L_{t+1}) - p_t) \mid E^c] = 0.$$