Giffen Demand for Several Goods

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Abstract

The utility maximizing consumer's demand function may simultaneously possess the Giffen property for any number of goods strictly less than all. By way of a simple example it is possible to illuminate the preference characteristics conducive to such a result.

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1 Introduction

Recent examples of simple, standard utility functions can be used to obtain a fuller understanding of the Giffen effect in standard demand theory.[1] The Giffen effect arises when a consumer's demand for a good is locally increasing in the good's own price. Microeconomics textbooks often mention that the demand function for one good may possess the Giffen property; that Giffen goods must be inferior; and that not all goods can be simultaneously inferior. By implication, not all goods can be simultaneously Giffen.[2]

Here, the converse statement is explored. Any number but one of the goods can simultaneously have the Giffen property. A particular price-income pair is held fixed, and the demand for each good in question is locally a strictly increasing function of its own price.[3]

Generalizing the main idea from Sørensen's (2007) two-good analysis, the present article considers a consumer with utility function $u(x) = \min\{u_1(x), \ldots, u_L(x)\}$, where $u_1, \ldots, u_L$ are themselves standard utility functions over L-good consumption bundles $x = (x_1, \ldots, x_L)$. The utility function u represents preferences for perfect complements in the intermediate utility indices $u_1, \ldots, u_L$. Bearing in mind that the most popular examples of utility functions (perfect substitutes, perfect complements, Cobb-Douglas, constant elasticity of substitution) derive the utility index as some weighted average of the consumed amounts $x_1, \ldots, x_L$, it is natural to compound the average of such averages.[4]

Along the lines of Sørensen (2007), the focus is on the budget situations where the consumer optimally demands a bundle $\hat{x}$ at an intersection of the L indifference surfaces, $u_1(\hat{x}) = \cdots = u_L(\hat{x}) = \hat{u}$. At such a corner of the indifference surface $u(x) = \hat{u}$, there is no substitution effect as the price changes. The consumer's reaction to a small price increase is then equivalent to the reaction to a small income reduction.[5] A good now has

[1] Heijman and von Mouche (2009) survey the literature.

[2] For completeness, a new version of this result is spelled out towards the end of this article.

[3] Holding fixed only the price vector, but allowing the income to change, different goods can have the Giffen property at different income levels. In this weaker sense it is possible for all goods to exhibit the Giffen property at some price vector. An explicit example appears in Example 5 of Sørensen (2004).

[4] This construction of utility functions has been suggested before, see e.g. exercise 8.13 of Varian (1992) and exercises 1.12(b) and 1.27 of Jehle and Reny (2001).

[5] Figure 17.E.1 of Mas-Colell et al. (1995) illustrates preferences of a very similar nature. The expansion

the local Giffen property if and only if it is locally inferior. The only remaining problem is then to place a backward-bending income expansion path, which is analytically simple.[6]

2 Utility

Let $J, L \in \mathbb{N}$. Consider a consumer with consumption set $\mathbb{R}_+^L$ whose utility function satisfies $u(x) = \min\{u_1(x), \ldots, u_J(x)\}$, where each u_j is a function from $\mathbb{R}_+^L$ to $\mathbb{R}$. The utility function u from $\mathbb{R}_+^L$ to $\mathbb{R}$ satisfies many usual properties if each of the component utility functions u_j does so. Here is a list of such properties (see also Section 3.B of Mas-Colell et al., 1995). Property (vi) is also known as convexity of the underlying preference relation.

(i) Continuity: u is a continuous function.

(ii) Monotonicity: if $y \gg x$, then $u(y) > u(x)$.

(iii) Strong Monotonicity: if $y \geq x$ and $y \neq x$ then $u(y) > u(x)$.

(iv) Weak Monotonicity: if $y \geq x$ then $u(y) \geq u(x)$.

(v) Concavity: u is a concave function.

(vi) Quasi-Concavity: for any $\bar{u} \in \mathbb{R}$, the upper contour set $\left\{x \in \mathbb{R}_+^L : u(x) \geq \bar{u}\right\}$ is convex.

(vii) Strict Quasi-Concavity: for any $\bar{u} \in \mathbb{R}$, any $x, y \in \mathbb{R}_+^L$ with $x \neq y$, and any $\alpha \in (0,1)$, if $u(x) \geq \bar{u}$ and $u(y) \geq \bar{u}$, then $u(\alpha x + (1-\alpha) y) > \bar{u}$.

(viii) Homogeneity of Degree 1: u maps $\mathbb{R}_+^L$ into $\mathbb{R}_+$, and $u(\alpha x) = \alpha u(x)$ for all $\alpha \in \mathbb{R}_+$.

It is easy to verify that each of those 8 properties, one by one, is inherited by u from $u_1, \ldots, u_J$:

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curves are parallel to the axes, providing a Giffen neutrality effect, where the demand for a good does not respond to a marginal change in its own price.

[6]constructing a utility function with Giffen demand, Moffatt (2002) proceeds in the reverse direction. He first takes the backward-bending income expansion path, and then attaches a string of nearly-kinked indifference curves to it.

Lemma 1 *Let $n \in \{i, \ldots, viii\}$. Suppose that $u_1, \ldots, u_J$ all satisfy property (n). Then u satisfies (n).*

Proof. (i) min is a continuous function, and continuity is preserved by the composition of functions. (ii) and (iii) Suppose y and x are as assumed in the property. By definition of u there exists some j such that $u(y) = u_j(y)$. Given that u_j satisfies the property, then $u(x) \leq u_j(x) < u_j(y) = u(y)$. (iv) Similar to (ii) and (iii), except that the assumption is $u_j(x) \leq u_j(y)$, which suffices for the conclusion. (v) Let $x, y \in \mathbb{R}^L_+$ and $\alpha \in [0,1]$, and verify Jensen's inequality: $u(\alpha x + (1-\alpha)y) = \min\{u_1(\alpha x + (1-\alpha)y), \ldots, u_J(\alpha x + (1-\alpha)y)\} \geq \min\{\alpha u_1(x) + (1-\alpha)u_1(y), \ldots, \alpha u_J(x) + (1-\alpha)u_J(y)\} \geq \alpha u(x) + (1-\alpha)u(y)$, where the first inequality uses the concavity of the u_j functions with min being increasing in its arguments, and the second uses concavity of min. (vi) Suppose that $x \in \mathbb{R}^L_+$ and $\bar{u} \in \mathbb{R}$. From the definition of u, $u(x) \geq \bar{u}$ if and only if for every j, $u_j(x) \geq \bar{u}$. Thus the upper contour set for u is the intersection of the J sets for $u_1, \ldots, u_J$. Since the intersection of convex sets is convex, (vi) follows. (vii) Suppose that $\bar{u}, x, y, \alpha$ are given as stated, and that $u(x), u(y) \geq \bar{u}$. By the definition of u, there exists some j such that $u(\alpha x + (1-\alpha)y) = u_j(\alpha x + (1-\alpha)y)$. As noticed in the proof of (vi), we must have $u_j(x), u_j(y) \geq \bar{u}$. Since u_j satisfies (vii), it follows that $u(\alpha x + (1-\alpha)y) = u_j(\alpha x + (1-\alpha)y) > \bar{u}$ as desired. (viii) We obtain $u(\alpha x) = \min\{u_1(\alpha x), \ldots, u_J(\alpha x)\} = \min\{\alpha u_1(x), \ldots, \alpha u_J(x)\} = \alpha \min\{u_1(x), \ldots, u_J(x)\} = \alpha u(x)$. ■

If the consumption set allows good 1 to vary throughout all of $\mathbb{R}$, a straightforward exercise proves that Lemma 1 also applies to this property of quasi-linearity with respect to good 1: $u(x + \alpha e_1) = u(x) + \alpha$ for all $\alpha \in \mathbb{R}$, where $e_1 = (1, 0, \ldots, 0)$.

Lemma 1 can be generalized to handle the minimum of an infinite family of utility functions. Suppose thus that $U(x) = \min_{j \in I} u(x, j)$ where I is a compact set and u is continuous. These assumptions guarantee that the minimum is achieved, but also imply that U is continuous by the Theorem of the Maximum. It is a straightforward exercise to see that the proof of Lemma 1 can be modified to show that the other 7 properties (ii),...,(viii) of $u(\cdot, j)$ are again inherited by U.

The following standard result will be useful in the remainder.

Lemma 2 *Suppose that u is a function from $\mathbb{R}_+^L$ to $\mathbb{R}$ satisfying (i) continuity, (ii) monotonicity and (vi) quasi-concavity. For any $x \in \mathbb{R}_+^L$, there exists $p \in \mathbb{R}_+^L$ such that $u(y) > u(x)$ implies $p \cdot y > p \cdot x$. If u is differentiable at x with $\nabla u(x) \neq 0$, then p must be proportional to $\nabla u(x)$.*

Proof. First, let $A = \left\{y \in \mathbb{R}_+^L | u(y) > u(x)\right\}$. Properties (i) and (vi) imply that A is open and convex, and $x \notin A$ by construction. A standard separation theorem for convex sets, such as 11.2 in Rockafellar (1970), yields the existence of $p \in \mathbb{R}^L$ with $p \cdot y > p \cdot x$ for all $y \in A$. Also, $p \geq 0$ is necessary since (ii) implies that $\{x\} + \mathbb{R}_{++}^L \subseteq A$. Second, suppose that u is differentiable at x with $\nabla u(x) \neq 0$. If p is not proportional to $\nabla u(x)$, then there exists $z \in \mathbb{R}^L$ with $p \cdot z < 0 < \nabla u(x) \cdot z$. By differentiability, there exists $\varepsilon > 0$ with $u(x + \varepsilon z) > u(x)$, so that $x + \varepsilon z \in A$. The separation result then implies $p \cdot (x + \varepsilon z) > p \cdot x$, in contradiction to $p \cdot z < 0$. ■

3 Demand

Suppose $L \geq 2$. Given income $m > 0$ and price vector $p = (p_1, \ldots, p_L) \in \mathbb{R}_{++}^L$, the consumer chooses $x = (x_1, \ldots, x_L)$ to maximize utility $u(x)$ subject to the budget constraint $p \cdot x = p_1 x_1 + \cdots + p_L x_L \leq m$. The maximand is the Marshallian demand $x(p, m)$.

The Marshallian demand for good ℓ possesses the Giffen property at price-income pair (p, m) if there is a strictly positive marginal response in $x_\ell(p, m)$ to a partial change in p_ℓ.

In the following, we will suppose that $J = L$ and that the consumer's utility function satisfies $u(x) = \min\{u_1(x), \ldots, u_L(x)\}$. We wish to study the consumer's demand function near a situation where the demanded bundle x sits at a kink of the indifference surface for u. Naturally, such a kink arises at x where $u_1(x) = \cdots = u_J(x)$ and the gradient vectors $\nabla u_1(x), \ldots, \nabla u_J(x)$ are linearly independent. Note that such a point x would not define a kink of the indifference surface if $J < L$, and that $J - L$ of the functions u_j would be redundant in the definition of the utility function u near x if $J > L$. Hence, the remainder of the text assumes $J = L$.

Proposition 3 *Suppose that $u_1, \ldots u_L$ all satisfy (i) continuity, (ii) monotonicity and (vi)* *quasi-concavity. Assume that $\hat{x} \in \mathbb{R}^L_{++}$ solves $u_1(\hat{x}) = \cdots = u_L(\hat{x})$, that $u_1, \ldots, u_L$ are C^1 at $\hat{x}$, and that the gradient vectors $\nabla u_1(\hat{x}), \ldots, \nabla u_L(\hat{x})$ are linearly independent. Take as given any price vector $\hat{p} = \lambda_1 \nabla u_1(\hat{x}) + \cdots + \lambda_L \nabla u_L(\hat{x})$ with $(\lambda_1, \ldots, \lambda_L) \gg 0$, and let $\hat{m} = \hat{p} \cdot \hat{x}$. Define $\hat{d} \in \mathbb{R}^L \setminus \{0\}$ by*

$$\hat{d} = \left[\begin{array}{ccc} 1 & \cdots & 1 \end{array}\right] \left[\begin{array}{c|c|c} \nabla u_1(\hat{x}) & \cdots & \nabla u_L(\hat{x}) \end{array}\right]^{-1}. \tag{1}$$

If $\hat{d}_\ell < 0$, then good ℓ is a Giffen good near $(\hat{p}, \hat{m})$ for the consumer with utility function u.

Proof. First, observe that $\hat{x}$ is a solution to the utility maximization problem given $(\hat{p}, \hat{m})$. Define $\hat{u} \in \mathbb{R}$ by $\hat{u} = u_1(\hat{x})$. By definitions, $\hat{x}$ is feasible in the problem and achieves utility level $\hat{u}$. Consider any $y \in \mathbb{R}^L_+$ with $u(y) > \hat{u}$. Observe that for any j, $u_j(y) \geq u(y) > \hat{u} = u_j(\hat{x})$. By Lemma 2 applied to u_j, then $\nabla u_j(\hat{x}) \cdot y > \nabla u_j(\hat{x}) \cdot \hat{x}$. Since every $\lambda_j > 0$, it follows that $\hat{p} \cdot y > \hat{p} \cdot \hat{x} = \hat{m}$, so any such y lies outside the budget set. Second, by the Theorem of the Maximum, the maximal utility that the consumer can achieve is a continuous function of (p, m). Since the utility functions u_j are C^1, and since the gradient vectors are linearly independent, the first part of the proof extends to an open region of price-income pairs (p, m) near $(\hat{p}, \hat{m})$, in which region a solution $x(p, m)$ near $\hat{x}$ to the utility maximization problem solves the equations $u_1(x(p, m)) = \cdots = u_L(x(p, m)) = \tilde{u}(p, m)$ for some $\tilde{u}(p, m)$ near $\hat{u}$. Third, an increase in the price of good ℓ will ceteris paribus decrease the utility level, so the demanded bundle x moves along the curve solving $u_1(x) = \cdots = u_L(x) = \tilde{u}$ for decreasing utility level $\tilde{u}$. The claim is then that x_ℓ is a decreasing function of $\tilde{u}$ when $\hat{d}_\ell < 0$. This follows from an application of the implicit function theorem to the L equations $u_1(x) = \tilde{u}, \ldots, u_L(x) = \tilde{u}$. ■

The proposition actually verifies a stronger property than Giffen's when more than one good is involved. If the entries in $\hat{d}$ are negative for all of some subset K of the goods, then holding fixed the income as well as prices of all other goods, the Marshallian demand for all K goods in the subset will increase when all their prices are simultaneously marginally increased.

The proposition exploits the zero substitution effect at a kink in the indifference curve.

This extreme can be relaxed through approximation. The function $\min\{u_1, \ldots, u_L\}$ is approximated by the constant elasticity of substitution (CES) function $(u_1^\rho + \cdots + u_L^\rho)^{1/\rho}$ as $\rho \to -\infty$. If $u_1, \ldots, u_L$ are concave functions of x, then also $u(x) = (u_1^\rho(x) + \cdots + u_L^\rho(x))^{1/\rho}$ is concave in x. The function u therefore represents continous, monotone, convex preferences. When $-\rho$ is sufficiently large, all indifference curves near $\hat{x}$ are sufficiently close to those of $\min\{u_1(x), \ldots, u_L(x)\}$, and the sign of $\hat{d}$ determines whether the good is Giffen also in the CES case.

4 Example

An example will illustrate how the proposition can be applied to construct demand functions with local Giffen behavior. Let $L \geq 2$. Suppose that each u_j is of the familiar Cobb-Douglas form

$$u_j(x_1, \ldots, x_L) = x_1^{a_{j1}} \cdots x_L^{a_{jL}}, \tag{2}$$

with strictly positive constants $a_{j1}, \ldots, a_{jL}$. This utility function satisfies (i) continuity, (ii) monotonicity and (vi) quasi-concavity. Specifically, we will suppose that the vectors $a_1, \ldots, a_L \in \mathbb{R}_+^L$ are defined by $a_j = (1, \ldots, 1, b, 1, \ldots, 1)/c_j$ with $b > 1$ in the j'th coordinate, and $c_j \geq b + L - 1$. Since $a_{j1} + \cdots + a_{jL} \leq 1$, the utility function u_j is concave.

For the sake of the example, focus will be on the consumption bundle $\hat{x} = (1, \ldots, 1)$. For $\hat{u} = 1$, we have $u_j(\hat{x}) = \hat{u}$, and u_j is C^1 at $\hat{x}$ with $\nabla u_j(\hat{x}) = a_j'$. The assumptions of the proposition are satisfied, so calculation of the vector $\hat{d}$ helps to determine which are Giffen goods at $\hat{x}$. It is simple to verify that

$$\left[\ \nabla u_1(\hat{x}) \ \ | \ \cdots \ | \ \ \nabla u_L(\hat{x})\ \right]^{-1} = G \tag{3}$$

where row j of matrix G is

$$g_j = \frac{c_j}{b(b + L - 2) - (L - 1)} [-1, \ldots, -1, b + L - 2, -1, \ldots, -1] \tag{4}$$

with $b + L - 2$ in the j'th coordinate. Note that $b > 1$ and $L \geq 2$ imply $b + L - 2 > 1$ as well as $b(b + L - 2) - (L - 1) > 0$.

Now, it follows from (1) that

$$\hat{d}_\ell = \sum_{j=1}^{L} g_{j\ell} = \frac{c_\ell\,(b+L-2) - \sum_{j\neq\ell} c_j}{b\,(b+L-2)-(L-1)}. \tag{5}$$

Since the denominator is positive, it should be clear that the sign of $\hat{d}_\ell$ is determined by the relative size of c_ℓ to $\sum_{j\neq\ell} c_j$. In particular, $\hat{d}_\ell < 0$ and ℓ is a local Giffen good around the price-income pairs defined in the proposition, when c_ℓ is sufficiently relatively small. The only assumption is that each such positive constant is at least $b+L-1$, so it is not hard to make all but one such constant relatively small (by making the last one very large).

For an illustration, let $1 \leq k < L$ and suppose that $b+L-1 \leq c_1 = \ldots = c_k < c_{k+1} = \ldots = c_L$. Each of the first k goods will simultaneously be local Giffen goods provided $c_1\,(b+L-2-k+1) < (L-k)\,c_L$. This is true when c_L is sufficiently large.

A natural interpretation of the utility function $\min\{u_1(x),\ldots,u_L(x)\}$ follows from Lancaster's (1966) new consumer theory, in which consumers care about characteristics produced by the purchased bundle. In the present context, think of the concave Cobb-Douglas functions as production functions.[7] The assumption that $b > 1$ means that factor ℓ is relatively essential for the production of characteristic ℓ. The illustration then provides the important insight that, given the existence of a kink point $\hat{x}$, good ℓ is more likely to be Giffen if c_ℓ is relatively large. This means that the production of characteristic ℓ offers a relatively high marginal return to all its inputs. The following intuitive reason can then explain the Giffen property. When the price of good ℓ rises, the production of all characteristics will be scaled back. Since characteristics are perfect complements, those with lower marginal returns can scale back their relatively essential factors a lot for a small reduction in the characteristic. In turn, this scaling back would give too great a reduction in the high-marginal return characteristic ℓ if it were not for an increase in its relatively essential factor ℓ.

[7] This interpretation does not permit the consumer to allocate inputs in different proportions across productive activities. Lancaster (1966) formulates the model both with and without such a possibility.

5 Converse

As already mentioned in the introduction, it is commonly accepted that there cannot exist a price-income pair at which all L goods are simultaneously Giffen. The standard derivation of this result goes via the Slutsky equation. The argument thus invokes an assumption of differentiability of the demand function which needs not always be satisfied. For the sake of completeness, here is a result derived from first principles.

The solution to the utility maximization problem must satisfy the following weak axiom of revealed preferences (WARP). Given two pairs $(p, m), (p', m') \in \mathbb{R}_{++}^{L+1}$, if $x(p, m) \neq x(p', m')$ and if $p \cdot x(p', m') \leq m$ then $p' \cdot x(p, m) > m'$. Interpreting this, $p \cdot x(p', m') \leq m$ implies that $x(p', m')$ is a feasible choice at (p, m), so that utility maximization reveals the preference relation $u(x(p, m)) > u(x(p', m'))$.[8]

For convenience, it will be assumed that the demand function is continuous. The only purpose of this assumption in the argument is to ensure that there is a neighborhood around the hypothetical L Giffen good point (p, m) in which the demand function is bounded away from zero in all coordinates. Continuity of the demand function can be derived as a necessary implication of properties (i) continuity and (vii) strict quasi-concavity of the utility function.

Proposition 4 *Suppose that the demand function is continuous. There does not exist any $(p, m) \in \mathbb{R}_{++}^{L+1}$ with the property that for all $\ell \in \{1, \ldots, L\}$, good ℓ is is Giffen in a neighborhood of (p, m).*

Proof. Suppose to the contrary that such $(p, m) \in \mathbb{R}_{++}^{L+1}$ exists. The proof will seek a contradiction. Denote $x^0 = x(p, m)$. Observe that $x^0 \gg 0$ since the non-negative consumption of every good will be strictly smaller when its own price falls. Observe also that $p \cdot x^0 = m$, for there would otherwise exist $\hat{p} = p + \varepsilon p_1$ close enough to p with $\varepsilon > 0$ such that $\hat{p} \cdot x^0 \leq m$, in contradiction to the WARP because $x(\hat{p}, m) \neq x^0$ and $p \cdot x(\hat{p}, m) \leq \hat{p} \cdot x(\hat{p}, m) \leq m$. Denote $\ell^1 = 1$. Choose $p^1 \in \mathbb{R}_{++}^{L}$ close to p such that $p_\ell^1 = p_\ell$ for all $\ell \neq \ell^1$, and $p_{\ell^1}^1 > p_{\ell^1}$. Denote $x^1 = x(p^1, m)$. Observe that $x_{\ell^1}^1 > x_{\ell^1}^0 \geq 0$

[8] See for instance chapters 1–3 in Mas-Colell, Whinston and Green (1995) for details.

since ℓ^1 is a local Giffen good. Also, $p \cdot x^1 < p^1 \cdot x^1 \leq m = p \cdot x^0$, ruling out $x^1 \geq x^0$. Hence, there exists some $\ell^2 \in \{2, \ldots, L\}$ with the property that $x^1_{\ell^2} < x^0_{\ell^2}$. Next, choose $p^2 \in \mathbb{R}^L_{++}$ such that $p^2_\ell = p_\ell$ for all $\ell \neq \ell^2$, and $\left(p^1_{\ell^1} - p_{\ell^1}\right) x^1_{\ell^1} = \left(p^2_{\ell^2} - p_{\ell^2}\right) x^1_{\ell^2}$. Denote $x^2 = x\left(p^2, m\right)$. Note that p^2 is arbitrarily close to p when p^1 was chosen close enough to p. By construction, $p^2 \cdot x^1 = p_{\ell^1} x^1_{\ell^1} + p^2_{\ell^2} x^1_{\ell^2} + \sum_{\ell \neq \ell^1, \ell^2} p_\ell x^1_\ell = p^1_{\ell^1} x^1_{\ell^1} + p_{\ell^2} x^1_{\ell^2} + \sum_{\ell \neq \ell^1, \ell^1} p_\ell x^1_\ell = p^1 \cdot x^1 \leq m$. Since ℓ^2 is a local Giffen good, $x^2_{\ell^2} > x^0_{\ell^2} > x^1_{\ell^2}$, so $x^2 \neq x^1$. By the WARP, we have $u\left(x^2\right) > u\left(x^1\right)$ and $p^1 \cdot x^2 > m \geq p^2 \cdot x^2$. Expanding this inequality, observe that $\left(p^1_{\ell^1} - p_{\ell^1}\right) x^2_{\ell^1} > \left(p^2_{\ell^2} - p_{\ell^2}\right) x^2_{\ell^2} > \left(p^2_{\ell^2} - p_{\ell^2}\right) x^1_{\ell^2} = \left(p^1_{\ell^1} - p_{\ell^1}\right) x^1_{\ell^1}$, implying $x^2_{\ell^1} > x^1_{\ell^1} > x^0_{\ell^1}$. Re-iterating the previous argument, $x^2 \geq x^0$ is impossible, so there must exist some $\ell^3 \neq \ell^1, \ell^2$ with $x^2_{\ell^3} < x^0_{\ell^3}$. Repeat this step inductively, until a final L-th iteration, in which $p^L \in \mathbb{R}^L_{++}$ is defined by $p^L_\ell = p_\ell$ for all $\ell \neq \ell^{L-1}$, and $\left(p^{L-1}_{\ell^{L-1}} - p_{\ell^{L-1}}\right) x^{L-1}_{\ell^{L-1}} = \left(p^L_{\ell^L} - p_{\ell^L}\right) x^{L-1}_{\ell^L}$. Continuity of the demand function implies that if the first choice of p^1 is sufficiently close to p, then every $p^2, \ldots, p^L$ constructed in this fashion is so close to p that all the Giffen properties hold. Again, $u\left(x^L\right) > u\left(x^{L-1}\right)$ by the WARP, implying $x^L_{\ell^{L-1}} > x^{L-1}_{\ell^{L-1}} > x^0_{\ell^{L-1}}$. The Giffen good property of L and the result of the previous induction step give $x^L_{\ell^L} > x^0_{\ell^L} > x^{L-1}_{\ell^L}$. Also, for every $z \in \{1, \ldots, L-1\}$, $u\left(x^L\right) > u\left(x^{L-z}\right)$ implies $p^{L-z} \cdot x^L > m = p^L \cdot x^L$. This is expanded as $\left(p^{L-z}_{\ell^{L-z}} - p_{\ell^{L-z}}\right) x^L_{\ell^{L-z}} > \left(p^L_{\ell^L} - p_{\ell^L}\right) x^L_{\ell^L} > \left(p^L_{\ell^L} - p_{\ell^L}\right) x^{L-1}_{\ell^L} = \left(p^{L-1}_{\ell^{L-1}} - p_{\ell^{L-1}}\right) x^{L-1}_{\ell^{L-1}} > \cdots > \left(p^{L-z}_{\ell^{L-z}} - p_{\ell^{L-z}}\right) x^{L-z}_{\ell^{L-z}}$. Hence, $x^L_{\ell^{L-z}} > x^{L-z}_{\ell^{L-z}} > x^0_{\ell^{L-z}}$ for every $z \in \{1, \ldots, L-1\}$. Since also $x^L_{\ell^L} > x^0_{\ell^L}$, it follows that $x^L \gg x^0$. This is the desired contradiction to $p \cdot x^L < p^L \cdot x^L = m = p \cdot x^0$. ■

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