CALCULATING THE TIME IT TAKES TO GET HALF WAY TO STEADY STATE IN THE SOLOW MODEL

Lecture note: macroeconomic 2
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We begin with our growth equation:

$$\frac{k_{t+1}}{k_t} - 1 = \frac{1}{1+n}\left[s\frac{f(k_t, A)}{k_t} - (\delta + n)\right] \equiv G(k_t)$$

Next, perform a 1st order Taylor approximation around steady state

$$\begin{aligned} G(k) &\approx G(k^*) + G'(k^*)(k_t - k^*) \\ &= 0 + \frac{s}{1+n}\left(\frac{f'(k;A)k - f(k;A)}{k^2}\right)(k_t - k^*) \\ &= \frac{sf(k^*)/k^*}{1+n}\left(\frac{f'(k^*;A)k^* - f(k^*;A)}{f(k^*)}\right)\left(\frac{k_t - k^*}{k^*}\right) \end{aligned}$$

Next, observe that $sf(k^*)/k^* = n + \delta$ in the steady state, and,

$$\frac{f'(k^*;A)k^* - f(k^*;A)}{f(k^*)} = \frac{r^*k^* - y^*}{y^*} = -[1 - \alpha(k^*)]$$

where $\alpha(k^*)$ therefore is the share of capital in national accounts in the steady state.

Accordingly, we have

$$\frac{k_{t+1}}{k_t} - 1 \approx -(1 - \alpha(k^*))\frac{n+\delta}{1+n}\left(\frac{k_t - k^*}{k^*}\right),$$

which (approximately) the same as saying:[1]

$$\ln k_{t+1} - \ln k_t \approx -\left\{[1 - \alpha(k^*)]\frac{n+\delta}{1+n}\right\}(\ln k_t - \ln k^*).$$

This is a useful intermediate result.

First, it makes it clear that the economy will be growing faster the farther away it is from its steady state (i.e., $\ln k_{t+1} - \ln k_t$ is increasing in the distance $\ln k^* - \ln k_t$). Second, it tells us that every time the capital stock increases by 1 percent, the growth rate declines by $[1 - \alpha(k^*)]\frac{(n+\delta)}{1+n}$ percent. This is

[1] In discrete time we have $x_{t+1} = (1+g)x_t$, where g is the growth rate of x. In log terms $\ln x_{t+1} = \ln(1+g) + \ln x_t$ which is approx. equal to $g + \ln x_t$ when g is small.

diminishing returns "in action". It is worth remarking, that *if we were employing a Cobb-Douglas production function,* i.e.

$$f(k; A) = Ak^{\alpha}$$

, the formula would be the same. Only $\alpha(k^*)$ would no longer depend on the factor intensity in the steady state (k^*), but would reduce to the exponent of the production function, α.

Anyway, we would like to know more. Specifically, we would like to figure out how *quickly* the economy is moving towards steady state. If we let $z_t \equiv \ln k_t - \ln k^*$, implying $z_{t+1} - z_t = \ln k_{t+1} - \ln k_t$, we have

$$z_{t+1} = \left(1 - [1 - \alpha(k^*)] \frac{n+\delta}{1+n}\right) z_t$$

a simple (linear) first order difference equation with the solution:

$$z_t = \left[1 - [1 - \alpha(k^*)] \frac{n+\delta}{1+n}\right]^t z_0$$

Re-inserting $\ln k_t - \ln k^*$, we have

$$\frac{\ln k_t - \ln k^*}{\ln k_0 - \ln k^*} = \left[1 - [1 - \alpha(k^*)] \frac{n+\delta}{1+n}\right]^t.$$

Now, this expression tells us the ratio of the distance left to steady state, $\ln k_t - \ln k^*$, relative to the total initial distance to steady state, $\ln k_0 - \ln k^*$. As $t \to \infty$ the ratio tends to zero (as $\frac{(1+n)-(1-\alpha)(n+\delta)}{1+n} < 1$).

As you can see, the model tells us that each period a fraction $\frac{(1+n)-(1-\alpha)(n+\delta)}{1+n}$ of the gap is eliminated.

More specifically, we may ask how much *time* it will take us to get half way to the steady state. The answer (by putting $\frac{\ln k_t - \ln k^*}{\ln k_0 - \ln k^*} = 1/2$, taking logs and rearranging):

$$t_{1/2} = \frac{-\ln(2)}{\ln\left(\frac{(1+n)-(1-\alpha)(n+\delta)}{1+n}\right)}$$

This expression is a bit messy, unfortunately. But we can insert plausible values for key parameters nevertheless. We choose $n = 0.01, \delta = 0.05$ and $\alpha = 0.4$. The result is

$$t_{1/2} = \frac{-\ln(2)}{\ln\left(\frac{(1+0.01)-(1-.4)(0.01+0.05)}{1+0.01}\right)} \approx 19.$$

Hence, under plausible assumptions the model predicts that it takes 19 years to close 50% of the gap to the steady state. Hence, the model *does* generate lengthy transitions.

A much more elegant formula can be derived, if we resort to continuos time. In fact, you can show that

$$t_{1/2} = \frac{\ln(2)}{(1-\alpha)(n+\delta)}.$$

If you use the same parameter values, you get virtually the same result - about 19 years (as you should, off course).

But it also provides a useful “rule-of-thumb”. Since $\log(2)$ is about 0.7, you can easily calculate the time it takes you to get half way to steady state by dividing that number by the rate of convergence in percent.

Finally, it is worth observing that in continuos time the *rate of convergence* simplifies to the expression

$$(1-\alpha)(n+\delta),$$

which is similar to, yet simpler than, the discret time counterpart:

$$(1-\alpha)\frac{n+\delta}{1+n}$$