Bank Assets, Liquidity and Credit Cycles: Technical Appendix

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Appendix A. Proofs

Proof of Proposition 1

As borrowers' marginal product of capital equals one in the steady state, we restrict our analysis to the impact of ξ on mpk^I:

$$\frac{\partial mpk^I}{\partial \xi} = \frac{\partial mpk^I}{\partial R^B} \frac{\partial R^B}{\partial \xi}. \tag{1}$$

As for the partial derivative of bankers' marginal product of capital with respect to the loan rate:

$$\frac{\partial mpk^I}{\partial R^B} = -\frac{\varkappa \omega \beta^B}{\kappa^2 R^S \beta^I}, \tag{2}$$

where $\kappa \equiv R^F \left(1 - \beta^F\right) - \omega \left(1 - \beta^F R^F\right) > 0$ and $\varkappa \equiv R^S \left(1 - \beta^I\right) - \chi \left(1 - \beta^I R^S\right) > 0$, so that $\partial mpk^I / \partial R^B < 0$.

As for $\partial R^B / \partial \xi < 0$, this is negative, in light of assuming $\beta^I R^S < 1$:

$$\frac{\partial R^B}{\partial \xi} = -\frac{\chi \left(1 - \beta^I R^S\right)}{\beta^I R^S}. \tag{3}$$

Thus, both factors on the right-hand side of (1) are negative and, since $\partial \Delta / \partial \xi = -\partial mpk^I / \partial \xi$, increasing ξ inevitably reduces the productivity gap.■

Proof of Proposition 2

We first prove that increasing ξ attenuates the impact of the technology shock on borrowers' capital-holdings. According to Equation (35) in the main text, v quantifies the pass-through of $\hat{\alpha}_t$ on $\hat{k}_t^B$. In turn, the marginal impact of ξ on v can be computed as:

$$\frac{\partial v}{\partial \xi} = \frac{(\lambda - \phi)(1 - \rho)\rho}{(1 - \lambda)(1 - \phi\rho)} \frac{\partial \eta}{\partial \xi} + \frac{(\lambda\rho - 1)(1 - \rho)\rho}{(1 - \phi\rho)^2} \frac{\partial \phi}{\partial \xi}, \tag{4}$$

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where:

$$\frac{\partial \eta}{\partial \xi}=\frac{\partial \eta}{\partial k^B}\frac{\partial k^B}{\partial R^B}\frac{\partial R^B}{\partial \xi} \quad \text{and} \quad \frac{\partial \phi}{\partial \xi}=\frac{\partial \phi}{\partial R^B}\frac{\partial R^B}{\partial \xi}. \tag{5}$$

Focusing on the second term on the right-hand side of (4), we can show this is negative, as: (i) $\frac{(\lambda\rho-1)(1-\rho)\rho}{(1-\phi\rho)^2}<0$, given that $\lambda\rho<1$; (ii) $\partial\phi/\partial R^B=-\omega/\left(R^B\right)^2<0$; (iii) $\partial R^B/\partial\xi<0$, as implied by (3).

As for the first term on the right-hand side of (4): $\frac{(\lambda-\phi)(1-\rho)\rho}{(1-\lambda)(1-\phi\rho)}>0$. Furthermore:

$$\frac{\partial \eta}{\partial k^B}=-\frac{1}{(1-\mu)\left(k^B\right)^2}<0$$

and

$$\frac{\partial k^B}{\partial R^B}=\frac{\omega}{\kappa R^B(\mu-1)}\left(\frac{1}{\mu}\frac{R^B\beta^B\varkappa}{R^S\beta^I\kappa}\right)^{\frac{1}{\mu-1}}<0, \tag{6}$$

where $\kappa\equiv R^B\left(1-\beta^B\right)-\omega\left(1-\beta^B R^B\right)$ and $\varkappa\equiv R^S\left(1-\beta^I\right)-\chi\left(1-\beta^I R^S\right)$. As $\partial R^B/\partial\xi<0$, also the first term on the right-hand side of (4) is negative. Therefore, ν is a negative function of ξ.

As for the impact of technology shocks on the capital price:

$$\frac{\partial \gamma}{\partial \xi}=\frac{\partial \gamma}{\partial \phi}\frac{\partial \phi}{\partial \xi}. \tag{7}$$

As for $\partial\gamma/\partial\phi$:

$$\frac{\partial \gamma}{\partial \phi}=-\frac{1-\rho}{(1-\phi\rho)^2}\rho<0, \tag{8}$$

while we already know that $\partial\phi/\partial\xi>0$. Therefore, the overall effect of ξ on γ is negative.■

Proof of Proposition 3

We know that $G'(k^I)$ is a decreasing function of θ. Thus, we aim to prove that the gap between bankers' and borrowers' marginal product of capital is greater than zero at $\theta=0$. To this end, we combine the capital Euler equations of bankers and borrowers, obtaining:

$$G'(k^I)\Big|_{\theta=0}=\frac{R^B\beta^B\left(R^S-1\right)}{\left(1-\beta^B\right)R^B-\omega\left(1-\beta^B R^B\right)}.$$

We then impose $G'(k^I)\Big|_{\theta=0}<1$ to obtain

$$R^B>\frac{\omega}{1-\beta^B\left(R^S-\omega\right)}.$$

As $R^B\Big|_{\theta=0}=\frac{R^S\left(1+\beta^I\right)-1}{\beta^I}$, all we need to prove is that

$$\frac{R^S\left(1+\beta^I\right)-1}{\beta^I}>\frac{\omega}{1-\beta^B\left(R^S-\omega\right)},$$

which can be manipulated to obtain

$$\left(1-\beta^B R^S\right)\left[\beta^I\left(R^S-\omega\right)+R^S-1\right]+\left(R^S-1\right)\beta^B\omega>0.$$

As $\beta^B R^S<1$, it is immediate to verify that both terms on the left-hand side of the last inequality are positive.

■

Appendix B. Equilibrium conditions (for online publication only)

Derivation of key equilibrium conditions

Borrowers maximize their utility under the collateral and the flow-of-funds constraints, taking R^B as given. The corresponding Lagrangian reads as:

$$\begin{aligned}\mathcal{L}_t^B &= E_0 \sum_{t=0}^{\infty} \left(\beta^B\right)^t \left\{c_t^B - \vartheta_t^B \left[c_t^B + R^B b_{t-1}^B + q_t(k_t^B - k_{t-1}^B) - b_t^B - \alpha_t k_{t-1}^B\right]\right. \\ &\quad \left. -\upsilon_t \left(b_t^B - \omega \frac{q_{t+1} k_t^B}{R^B}\right)\right\}, \end{aligned} \tag{9}$$

where ϑ_t^B and υ_t are the multipliers associated with borrowers' budget and collateral constraint, respectively. The first-order conditions are:

$$\frac{\partial \mathcal{L}_t^B}{\partial b_t^B} = 0 \Rightarrow -\beta^B R^B E_t \vartheta_{t+1}^B + \vartheta_t^B - \upsilon_t = 0; \tag{10}$$

$$\frac{\partial \mathcal{L}_t^B}{\partial k_t^B} = 0 \Rightarrow -\vartheta_t^B q_t + \beta^B E_t \left[\vartheta_{t+1}^B q_{t+1}\right] + \beta^B E_t \left[\vartheta_{t+1}^B \alpha_{t+1}\right] + \omega \upsilon_t E_t \left[\frac{q_{t+1}}{R^B}\right] = 0. \tag{11}$$

Condition (10) implies that a marginal decrease in borrowing today expands next period's utility and relaxes the current period's borrowing constraint. As for (11), acquiring an additional unit of capital today allows to expand future consumption not only through the conventional capital gain and dividend channels, but also through the feedback effect of the expected collateral value on the price of capital. As we consider linear preferences (i.e., $\vartheta_t^B = \vartheta^B = 1$), (10) implies $\upsilon_t = \upsilon = 1 - \beta^B R^B$.[1] Thus, the collateral constraint binds in the neighborhood of the steady state as long as $R^B < 1/\beta^B$, which is imposed throughout the rest of the analysis. Finally, (11) can be rewritten as

$$q_t = \frac{\beta^B R^B + \omega\left(1 - \beta^B R^B\right)}{R^B} E_t q_{t+1} + \beta^B E_t \alpha_{t+1}. \tag{12}$$

The Lagrangian for bankers' optimization reads, instead, as

$$\begin{aligned}\mathcal{L}_t^I &= E_0 \sum_{t=0}^{\infty} \left(\beta^I\right)^t \left\{c_t^I - \vartheta_t^I [c_t^I + R^S b_{t-1}^S + b_t^B + q_t(k_t^I - k_{t-1}^I)\right. \\ &\quad \left. -b_t^S - R^B b_{t-1}^B - \alpha_t G(k_{t-1}^I)] - \delta_t \left(b_t^S - \chi \frac{q_{t+1}}{R^S} k_t^I - \chi \xi \frac{b_t^B}{R^S}\right)\right\}, \end{aligned} \tag{13}$$

where ϑ_t^I and δ_t are the multipliers associated with bankers' budget constraint and enforcement constraint, respectively. The first-order conditions are:

$$\frac{\partial \mathcal{L}_t^I}{\partial b_t^S} = 0 \Rightarrow -R^S \beta^I E_t \vartheta_{t+1}^I + \vartheta_t^I - \delta_t = 0; \tag{14}$$

$$\frac{\partial \mathcal{L}_t^I}{\partial b_t^B} = 0 \Rightarrow R^B \beta^I E_t \vartheta_{t+1}^I - \vartheta_t^I + \frac{1}{R^S} \chi \xi \delta_t = 0; \tag{15}$$

$$\frac{\partial \mathcal{L}_t^I}{\partial k_t^I} = 0 \Rightarrow -\vartheta_t^I q_t + \beta^I E_t \left[\vartheta_{t+1}^I q_{t+1}\right] + \beta^I E_t \left[\vartheta_{t+1}^I \alpha_{t+1} G'\left(k_t^I\right)\right] + \delta_t \chi \frac{E_t \left[q_{t+1}\right]}{R^S} = 0. \tag{16}$$

[1] Steady-state variables are reported without the time subscript.

As we assume linear preferences, $\vartheta_t^I = \vartheta^I = 1$. Therefore, conditions (14) and (15) imply that the financial constraint holds with equality in the neighborhood of the steady state (i.e., $\delta_t = \delta > 0$) as long as (i) $R^S\beta^I < 1$ and (ii) $R^B\beta^I < 1$. By combining (14) and (15) we obtain

$$R^B = \frac{R^S - \chi\xi\left(1 - \beta^I R^S\right)}{\beta^I R^S}, \tag{17}$$

Finally, from (16) we can retrieve the Euler equation governing bankers' investment in real assets:

$$q_t = \frac{R^S\beta^I + \chi\left(1 - \beta^I R^S\right)}{R^S} E_t q_{t+1} + \beta^I E_t\left[\alpha_{t+1} G'\left(k_t^I\right)\right]. \tag{18}$$

Summary of the model

We have 12 endogenous variables: $\{q_t\}_{t=0}^{\infty}$, $\left\{c_t^S, b_t^S\right\}_{t=0}^{\infty}$, $\left\{c_t^B, b_t^B, k_t^B, y_t^B\right\}_{t=0}^{\infty}$, $\left\{c_t^I, b_t^I, b_t^S, k_t^I, y_t^I\right\}_{t=0}^{\infty}$, along with the aggregate productivity shifter: $\{\alpha_t\}_{t=0}^{\infty}$. The general equilibrium is characterized by the following equations:

- Market clearing (goods, credit and capital market, respectively):

$$\epsilon^S + y_t^B + y_t^I = \underbrace{c_t^S + c_t^B + c_t^I}_{y_t}, \tag{19}$$

$$b_t^S = \frac{\chi\xi}{R^S}\left(\omega\frac{k_t^B}{R^B} + \frac{1}{\xi}k_t^I\right) E_t q_{t+1}, \tag{20}$$

$$k_t^B + k_t^I = 1. \tag{21}$$

- Production technologies:

$$y_t^B = \alpha_t k_{t-1}^B, \tag{22}$$

$$y_t^I = \alpha_t G\left(k_{t-1}^I\right), \tag{23}$$

where

$$\log\alpha_t = \rho\log\alpha_{t-1} + u_t. \tag{24}$$

- Credit demand and supply:

$$b_t^B = \omega\frac{\mathbb{E}_t q_{t+1}}{R^B}, \tag{25}$$

$$b_t^B = \frac{1}{\chi\xi}\left(R^S b_t^S - \chi\mathbb{E}_t q_{t+1} k_t^I\right). \tag{26}$$

- Capital demand schedules:

$$q_t - \frac{\beta^I R^S + \chi\left(1-\beta^I R^S\right)}{R^S}\mathbb{E}_t q_{t+1} = \beta^I \mathbb{E}_t \alpha_{t+1} G'\left(k_t^I\right), \tag{27}$$

$$q_t - \left[\beta^B + \omega\left(\frac{1}{R^B} - \beta^B\right)\right]\mathbb{E}_t q_{t+1} = \beta^B \mathbb{E}_t \alpha_{t+1}. \tag{28}$$

- Budget constraints:

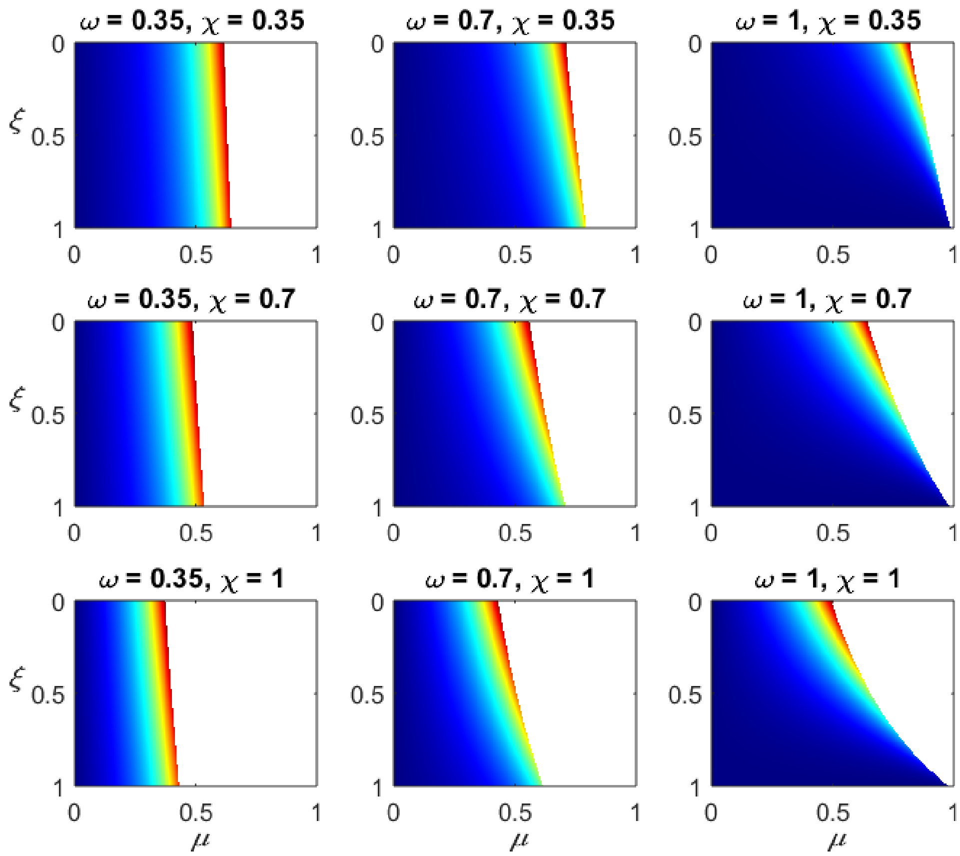
$$c_t^S + b_t^S = \epsilon^S + R^S b_{t-1}^S, \tag{29}$$

$$c_t^B + q_t k_t^B = y_t^B + q_t k_{t-1}^B + b_t^B - R^B b_{t-1}^B, \tag{30}$$

$$c_t^I + b_t^B + R^S b_{t-1}^S + q_t\left(k_t^I - k_{t-1}^I\right) = y_t^I + b_t^S + R^B b_{t-1}^B. \tag{31}$$

Appendix C. Robustness exercises (for online publication only)

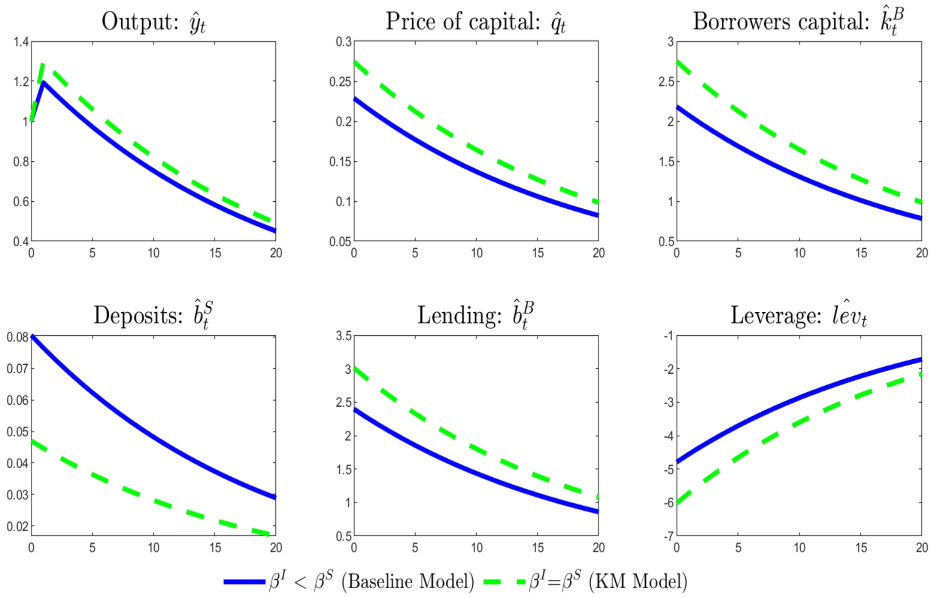
Figure C.1 Business cycle amplification.



Notes. Figure C.1 graphs ϖ as a function of ξ (y-axis) and μ (x-axis), and for different values of χ and ω, under the following parameterization: $\beta^S = 0.99$, $\beta^I = 0.98$, $\beta^B = 0.97$, $\rho = 0.95$. The white area denotes inadmissible equilibria where bankers' capital-holdings are virtually negative.

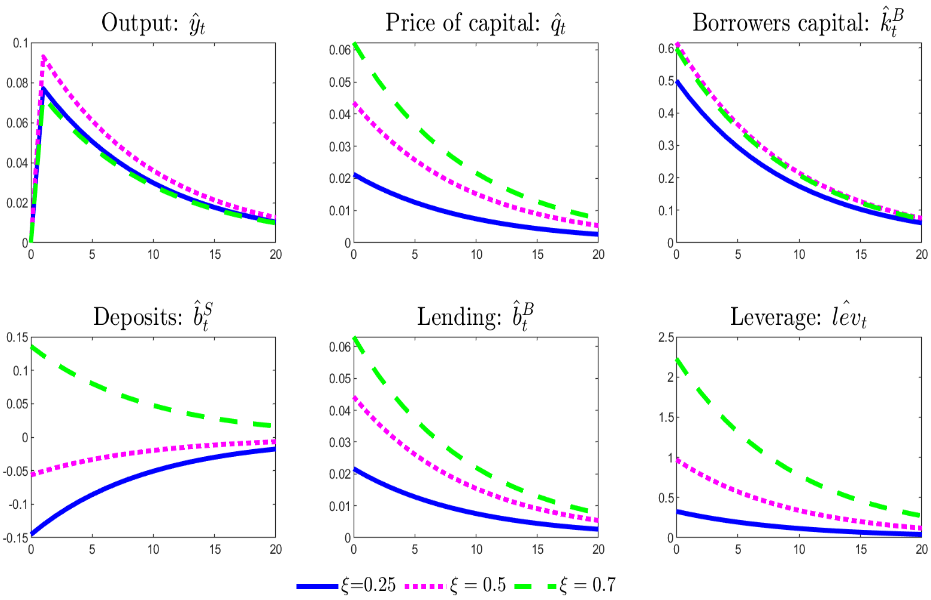
Appendix D. Additional figures (for online publication only)

Figure D1: Comparison with KM under a technology shock.



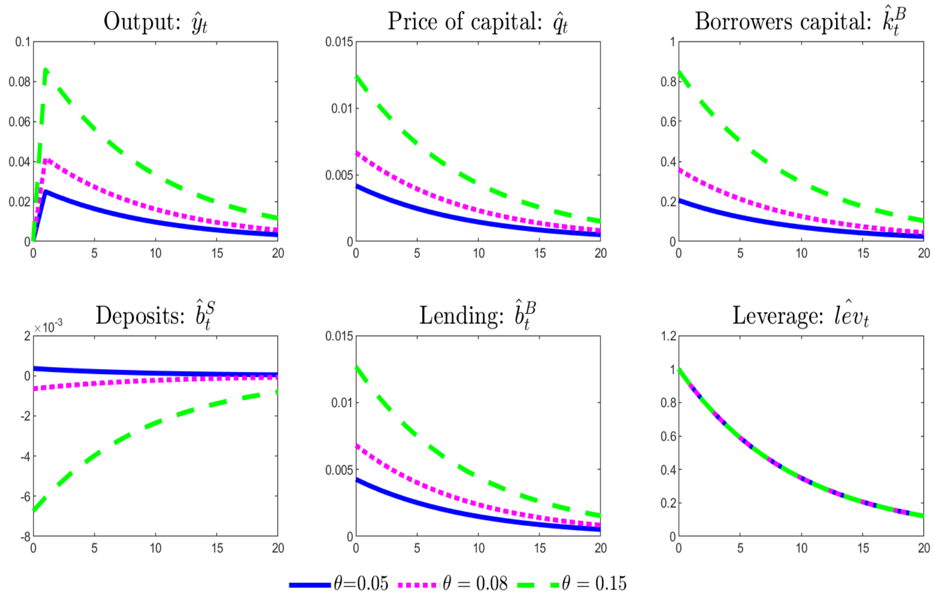
Notes. Figure D1 graphs the response to a one-standard-deviation shock to technology, under the following parameterization: $\beta^S = 0.99$, $\beta^I = 0.98$, $\beta^B = 0.97$, $\rho = 0.95$, $\chi = \omega = 1$, $\mu = 0.4$. We consider two situations: the KM case, where $1 - \beta^I R^S = 0$ (green-dashed line), and the baseline model (blue-continuos line).

Figure D2: Responses to a financial shock.



Notes. Figure D2 graphs the responses of selected variables to a one-standard-deviation shock to the degree of collateralization, ξ, under the following parameterization: $\beta^S = 0.99$, $\beta^I = 0.98$, $\beta^B = 0.97$, $\rho = 0.95$, $\chi = \omega = 1$, $\mu = 0.4$.

Figure D3: Responses to a shock to the capital-to-asset ratio.



Notes. Figure D3 graphs the responses of selected variables to a (negative) one-standard-deviation shock to capital-to-asset ratio, θ, under the following parameterization: $\beta^S = 0.99$, $\beta^I = 0.98$, $\beta^B = 0.97$, $\rho = 0.95$, $\omega = 1$, $\mu = 0.4$.