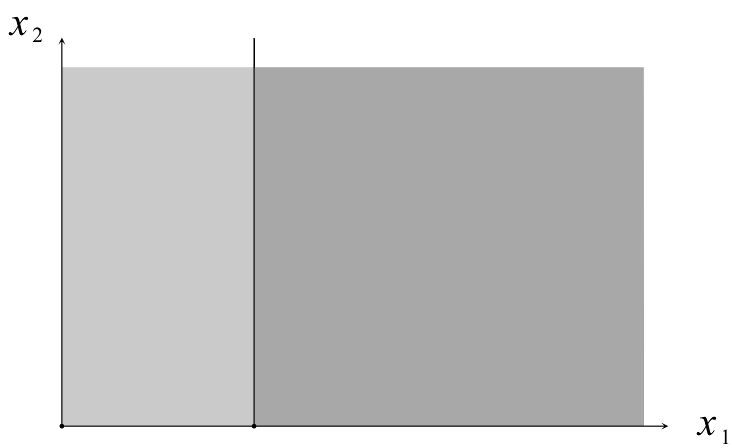
Exercises to Chapter 0

(1) The consumption set originally consists of all points in the darkly shaded set together with the x_1-axis, given that the consumer can use all bundles (x_1, x_2) yielding an initial amount $\overline{x}_1$ allowing the consumer to survive to the next period, where any amount x_2 is possible.



This consumption set is not convex, but allowing also combinations (x_1, x_2) where the consumer does not survive, we obtain a convex consumption set. The interpretation of a "feasible" consumption bundle is different, however.

(2) The relation $\succsim$ defined from P as p.6 is a complete preorder. By monotonicity, for each $x \in \mathbb{R}^l_+$, there is some $u > 0$ such that $ue \in P(x)$, where $e = (1, 1, \ldots, 1)$. Define the function $u : \mathbb{R}^l_+ \to \mathbb{R}$ by

$$u(x) = \inf\{u > 0 \mid ue \in P(x)\}$$

Using transitivity we find that $x \sim u(x)e$ and $u(x') = u(x) \Leftrightarrow z \sim x'$. If $x \succsim x'$, then $u(x)e \succsim u(x')e$, so that $u(x) \geq u(x')$, and conversely, $u(x) \geq u(x')$ implies that $u(x)e \succsim u(x')e$, so that $x \succsim x'$. We have therefore that u describes $\succsim$.

It remains to check that u is continuous, but this follows easily since for each x, $\{x' \mid x \succsim x\} = \operatorname{cl} P(u(x)e)$, which is a closed set, and $\{x' \mid x \succsim x'\} = \mathbb{R}^l_+ \backslash P(u(x)x)$ which is also open.

(3) The demand $\xi(p, w)$ is defined as the set of bundles x satisfying $p \cdot x \leq w$ such that $x' \in P(x)$ implies $p \cdot x' > w$. Since $\lambda p \cdot x'' \leq \lambda w$ exactly when $p \cdot x'' \leq w$ it is obvious that $\xi(\lambda p, \lambda w) = \xi(p, w)$.

Suppose now that $P(\lambda x) = \lambda P(x)$ for all $x \in \mathbb{R}^l_+$ and $\lambda > 0$ (preferences are homothetic). Let $x \in \xi(p,w)$ and $\lambda > 0$ be arbitrary. Then

$$p \cdot \lambda x = \lambda(p \cdot x) \leq \lambda w.$$

If $x' \in P(\lambda x)$, then $\lambda^{-1}x' \in P(x)$, so that $p \cdot (\lambda^{-}1x') > w$ or $p \cdot x' > \lambda w$, meaning that λx belongs to $\xi(p, \lambda w)$. Repeating the argument with $\bar{x} \in \xi(p, \lambda w)$ and λ^{-1}, we get that $x \in \xi(p,w) \Leftrightarrow \lambda x \in \xi(p, \lambda w)$.

(4) We check (i)-(iii) of Assumption 0.2. (i) is satisfied since elements x' of $P(x)$ must satisfy $x'_k > x_k$ for some $k \in \{1, \ldots, l\}$. For (ii), we notice that if $x'_k > x_k$ for all $k \in \{1, \ldots, l\}$, then $x' \in P(x)$, which therefore must be nonempty. Finally, $P(x)$ is convex: Let $x', x'' \in P(x)$ and $0 < \lambda < 1$. Let h be the smallest index such that either x'_h or x''_h is different from x_h. Then $\lambda x'_h + (1 - \lambda)x''_h > 0$, so that $\lambda x' + (1 - \lambda)x'' \in P(x)$ and also (iii) is satisfied.

The lexicographic preference P cannot be described by a continuous utility function: It is complete and transitive, but it fails to be continuous, indeed, the set $P(x)$ is in general not an open set in $\mathbb{R}^l_+$ (if $x_1 > 0$ then $P(x)$ contains points x' with $x'_1 = x_1$ and $x'_2 = 2x_2$, but any neighborhood of x' will contain points with $x''_1 < x_1$ which are not in P/x).

(5) The budget set $\gamma(p,w) = \{x \in \mathbb{R}_+ \mid p \cdot x \leq w\}$ is nonempty since $w \geq \min\{p \cdot x \leq w\}$, and it is closed and bounded, hence compact. Let u be a continuous utility function representing P. By Weierstrass' theorem, u attains its maximum at some point $x \in \gamma(p,w)$.

(6) For any $\lambda > 0$ and $y \in Y$, we have that

$$\prod_{k=2}^{l} (-\lambda y_k)^{\alpha_k} = \lambda^{\sum_{k=2}^{l} \alpha_k} \prod_{k=2}^{l} (-y_k)^{\alpha_k}.$$

Since

$$\lambda^\alpha = \lambda \text{ for } \alpha = 1,$$
$$\lambda^\alpha > \lambda \text{ for } \begin{cases} \alpha > 1, \lambda > 1 & \text{and} \\ \alpha < 1, \lambda < 1, \end{cases}$$

we get that Y has constant returns to scale for $\sum_{k=2}^{l} = 1$, increasing returns to scale for $\sum_{k=2}^{l} > 1$ and decreasing returns to scale for $\sum_{k=2}^{l} < 1$.

To find the profit maximizing production plan for some $p \in \mathbb{R}^l_{++}$, we need to maximize the profit function $\sum_{k=1}^{l} p_k y_k$ subject to the constraint $y_1 = (-y_2)^{\alpha_2} \cdots (-y_k)^{\alpha_k}$.

with Lagrangian

$$\sum_{k=1}^{l} p_k y_k - \lambda(y_1 - (-y_2)^{\alpha_2} \cdots (-y_k)^{\alpha_k}).$$

First order conditions are:

$$p_1 = \lambda,$$
$$p_k = \lambda \alpha_k \frac{y_1}{-y_k}, k = 2, \ldots, l$$

from which we get that the profit maximizing production is characterized by the equations

$$\frac{y_1}{-y_k} = \alpha_k \frac{p_1}{p_k},\ k = 2, \ldots, l,$$
$$\frac{y_h}{y_k} = \frac{\alpha_k}{\alpha_h} \frac{p_k}{p_h},\ h, k = 2, \ldots, k.$$

(7) If y^0 maximizes profits on Y, then $p \cdot y^0 > 0$ and constant returns to gale would imply that

$$p \cdot (\lambda y^0) = \lambda p \cdot y^0 > p \cdot y^0$$

for $\lambda > 1$, a contradiction. We conclude that $p \cdot y^0 = 0$.

(8) Let $y \in Y$ and let $\lambda \in \mathbf{R}_+$ be arbitrary. Choose $n \in \mathbf{N}$ such that $n \geq \lambda$, so that $\lambda = \mu n$ for some $\mu \in [0, 1]$. Using that $Y + Y \subset Y$ repeatedly, we get that

$$ny = \underbrace{y + \cdots + y}_{n \text{ times}} \in Y.$$

Applying that $0 \in Y$ and Y is convex we get that $\lambda y = \mu(ny) + (1.\mu)0 \in Y$.