Exercises to Chapter 1

(1) Assume that $q_i \in \Delta$, $i = 1, 2$. In the special case $q_1 = q_2 = q$, then any point x^* on the intersection of the line

$$\{(x_1, x_2) \mid q \cdot x = q \cdot \omega_1$$

with the Edgeworth box corresponds defines a Walras equilibrium.

Assume that $q_1 \neq q_2$. With linear preferences, consumer demand must be on the boundary of the consumption set, so that only the two corners of the box, where each commodity is used only by one consumer, can be equilibrium allocations. For the north-west corner, the equilibrium price p must be such that $q_{12} \geq p_2 \geq q_{22}$, so that the endowment of consumer 1 must belong to the cone spanned by the two lines through the corner defined by q_1 and q_2, and for the south-east corner, a similar argument shows that it may occur as the equilibrium allocation at prices p such that $q_{11} \geq p_1 \geq q_{21}$ when the endowment of consumer 1 is in the cone spanned by the two lines through this corner.

If the endowment point is not in any of the two cones, there is no Walras equilibrium.

(2) We find the demand of the two consumers at the price $(1, p)$: For consumer 1, we get the Lagrangian

$$\ln x_{11} + x_{12} + \lambda(4 + 6p - x_{11} - p x_{12}),$$

first order conditions give $\lambda = 1/p$ and $x_{11} = 1$, from which we get that $x_12 = \dfrac{3+6p}{p} = \dfrac{3}{p} + 6$. For consumer 2, we have a Lagrangian

$$x_{21} + 2\sqrt{x_2 2} + \lambda(6 + 4p - x_{21} - 2x_{22})$$

with first order conditions $\lambda = 1$ and $x_{22} = \dfrac{1}{p^2}$, so that $x_{12} = 6 + 4p - \dfrac{1}{p^2}$. Setting aggregate demand equal to total endowment, we get the equations

$$\begin{aligned} 7 + 4p - \frac{1}{p^2} &= 10 \\ \frac{3}{p} + 6 + \frac{1}{p^2} &= 10 \end{aligned} \tag{1}$$

Adding the two equations and multiplying by p, we get that the second order equation

$$4p^2 - 7p + 3 = 0$$

with roots $\frac{6}{8}$ and 1. It is checked that (1) is satisfied by $p = 1$, and we have a Walras equilibrium $((1, 9), (9, 1), (1, 1))$.

(3) At the price system $p = (p_1, p_2) \in \Delta$, we have the following consumer demands:

Consumer 1 maximizes $\min\{x_{11}, x_{12}\}$ subject to $p_1(x_{11} - 1) + p_2 x_{12} = p_1 + 2p_2$, with solution

$$(x_{11}, x_{12}) = (2 - p_1, 2 - p_1),$$

Consumer 2 maximizes $x_{21}^2 + x_{22}^2$ subject to $p_1 x_{21}^2 + p_2 x_{22} = 2p_1 + p_2$ with solution

$$(x_{21}, x_{22}) = \begin{cases} \left(0, 2\frac{p_1}{p_2} + 1\right) & p_1 \geq p_2 \\ \left(2 + \frac{p_2}{p_1}, 0\right) & p_1 \leq p_2 \end{cases}$$

for $p_1 \neq 0, p_2 \neq 0$. Aggregate demand is $\left(2 - p_1, 2\frac{p_1}{p_2} + 1 + 2 - p_1\right)$ if $p_1 \geq p_2$, and since aggregate endowment of both commodities is 3, we get that

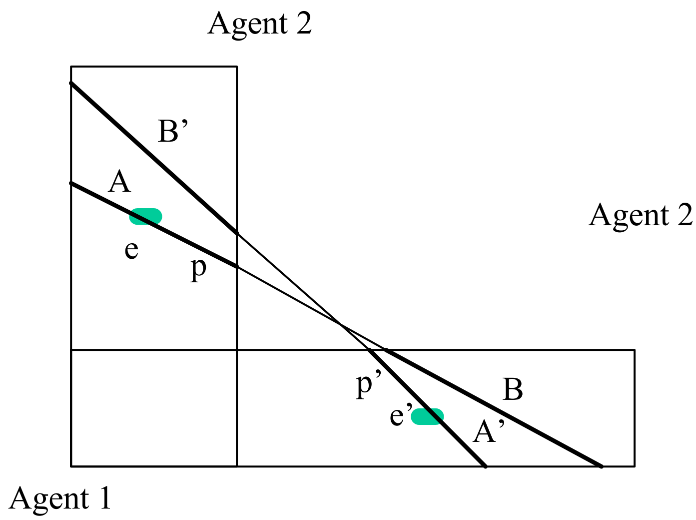
$$2\frac{p_1}{p_2} \leq p_1$$

which has no solution. Similarly, if $p_2 \geq p_1$, aggregate demand is $\left(2 - p_1 + 2 + 2\frac{p_2}{p_1}, 2 - p_1\right)$, and the inequality

$$2\frac{p_2}{p_1} \leq -1$$

has no solution $(p_1, p_2) \in \Delta$.

(4) We choose $l = 2$ and illustrate the two price-endowment pairs in the figure below (from: Levin, J. (2006), General Equilibrium).



If p is a Walras equilibirum price in the economy with the tall Edgeworth box and endowments e, then there is some point on the line segment A which is at least as good as everything on the budget line through e for agent 1. Using monotonicity, we get that there is a point on A which is better than everything on A'. Repeating the argument for the other Edgeworth box, we get a contradiction.

(5) Let (x, p) be a quasi-equilibrium, and suppose that i_0 satisfies the minimum wealth condition. Choose $i_1 \neq i_0$ arbitrarily. Since i_0 is resource related to i_1 at (x, p), there is an allocation $x' = (x'_1, \ldots, x'_m)$ and a bundle x''_{i_1} such that

$$\begin{gathered} \sum_{i=1}^{m}(x_i - \omega_i) + (x_{i_1} - \omega_{i_1}) = 0 \\ x'_{i_0} \in P_{i_0}(x_{i_0}), x'_i \in \operatorname{cl} P_i(x_i), \text{ all } i. \end{gathered} \tag{2}$$

Given that i_0 satisfies minimum-wealth at (x, p), we have that $p \cdot x'_{i_0} > p \cdot \omega_{i_0}$, and by the quasi-equilibrium properties, we get that $\sum_{i=1}^{m} p \cdot (x'_i - \omega_i) > 0$, and from (2) we get that

$$p \cdot x''_{i_1} < p \cdot \omega_{i_1},$$

showing that the minimum-wealth condition holds also for $_1$. Since i_1 was chosen arbitrarily, we get the result.

(6) If $x = (x_1, \ldots, x_m)$ maximizes $(p^1 \cdot x_1)^{\theta_1} \cdots (p^m \cdot x_m)^{\theta_m}$ under the constraint $\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i$, then we have a Lagrangian

$$(p^1 \cdot x_1)^{\theta_1} \cdots (p^m \cdot x_m)^{\theta_m} - \sum_{h=1}^{l} \lambda_h \sum_{i \in M} (\omega_{ih} - x_{ih})$$

with first order conditions

$$S\frac{\theta_i}{p^i \cdot x_i}p^i_h - \lambda_h = 0, i = 1, \ldots, m, h = 1, \ldots, l,$$

with $S = (p^1 \cdot x_1)^{\theta_1} \cdots (p^m \cdot x_m)^{\theta_m}$, so that

$$\frac{\theta_i p^i_h}{p^i \cdot x_i} = \frac{\lambda}{S} = p_h$$

is independent of i.

We have then for each i that

$$\sum_{h=1}^{l} p_h x_{ih} = \sum_{h=1}^{l}\left(\theta_i \frac{p^i_h}{p^i \cdot x_{ih}}\right)x_{ih} = \frac{\theta_i}{p^i \cdot x_i}\sum_{h=1}^{l} p^i_h x_{ih} = \theta_i,$$

and if $x'_i \in P_i(x_i)$, that is $p^i \cdot x'_i > p^i \cdot x_i$, then

$$p \cdot x'_i = \sum_{h=1}^{l}\left(\theta_i \frac{p^i_h}{p^i \cdot x_{ih}}\right)x'_{ih} = \theta_i \frac{p^i \cdot x'_i}{p^i \cdot x_i} > \theta_i,$$

showing that properties (i) and (ii) are satisfied.

(7) Suppose w.l.o.g. that φ_i is uhc with nonempty, closed and convex values for $i = 1, \ldots, r_1$, and φ_i has open graph and convex, possibly empty values for $i = r_1+1, \ldots, r_2$. For each of the correspondences φ_i which are convex-valued with open graph, we define a uhc corrrespondence $\hat{\varphi}_i : K \rightrightarrows K_i$ as follows: Let $D_i = \{x \in K : \varphi_1(x) \neq \emptyset\}$, and for each $x \in D_i$ choose some $\hat{x}_i \in \varphi_i(x)$ and a neighborhood U_x of x such that $\hat{x} \in \varphi_i(x')$ for $x' \in U_x$. The family $(U_x)_{x \in D_i}$ is an open covering of the paracompact space D_i, so that it has a point-finite refinement $\mathcal{U}$ with a subordinated continuous partition of unity $(\psi_U)_{U \in \mathcal{U}}$. It follows that the function $f_i : D_i \to K_i$ defined by

$$f_i(x) = \sum_{U \in \mathcal{U} : x \in U} \psi_U(x)\hat{x}_U,$$

where $\hat{x}_U$ is the element of K_i corresponding to U (or its relevant superset), is continuous, so that the correspondence $\hat{\phi}_i$ defined by

$$\hat{\phi}_i(x) = \begin{cases} \{f_i(x) & i \in D_i \\ K_i & x \notin D_i \end{cases}$$

Now we may apply Kakutani's fixed point theorem to the correspondence

$$varphi_1, \times \cdots \times \varphi_{r_1} \times \hat{\varphi}_{r_1+1} \times \cdots \times \hat{\varphi}_{r_2}$$

to obtain a point x^* such that

$$\begin{aligned} x_i^* &\in \varphi_i(x^*) \text{ for } i = 1, \ldots, x_{r_1} \\ x_i^* &\in \hat{\varphi}_i(x^*) \text{ for } i = r_1 + 1, \ldots, r_2. \end{aligned}$$

It is easily verified that x^* has the desired properties.

(8) Choose K such that $x_{i,h} < K$ for any feasible allocation-price pair (x, p). Let φ_i for $i = 0, 1, \ldots, m$ be the correspondences defined in the proof of Theorem 1.1 (p.30), and let $\hat{\varphi}_i$ be defined by

$$\hat{\varphi}_i(x) = \begin{cases} \varphi(x) & \varphi_i(x) \neq \emptyset \\ \{x' \in \mathbb{R}^l_+ \mid x_h \leq K, \text{ all } h\} & \text{otherwise} \end{cases}.$$

Define the correspondence P from $\{v \in \mathbb{R}^{(m+1)l}_+ \mid v_h \leq K, \text{ all } h\}$ to itself by

$$P(x_1, \ldots, x_m, p) = \begin{cases} \{v \in \mathbb{R}^{(m+1)l}_+ \mid v_h \leq K, \text{ all } h, v \neq 0\} & p = 0, \\ \emptyset & \varphi_i(x) = \emptyset, i = 0, 1, \ldots, m, \\ \hat{\varphi}_1(x, p) \times \cdots \times \hat{\varphi}_m(x, p) \times \hat{\varphi}_0(x, p) & \text{otherwise.} \end{cases}$$

Then it is easily checked that $P(x, p) = \emptyset$ exactly when (x, p) is a Walras equilibrium. Moreover, P is irreflexive with convex values and P has open graph.

Conversely, if P is a correspondence satisfying the condition, then $P(x, p) = \emptyset$ exactly when $\phi_i(x) = \emptyset$ for $i = 1, \ldots, m$ and $\phi_0(x, p) = \emptyset$, and otherwise there are no restrictions on P.