Exercises to Chapter 10

(1) Since the externality is an effect on the consumer's utility, for which no payment is given in the market equilibrium, this latter should be adapted so that payment is delivered. This can be done either by a Pigouvian tax t on commodity 1, so that the price paid by the producer and received by the consumer is $p_1 - t$. We normalize prices by $p_2 = 1$. Maximizing utility under the budget constraint

$$(t - p_1)(x_1 - c) + x_2 = 0$$

we get that $\frac{u_1'(x_1)}{u_2'(x_2)} = t - p_1$. The producer maximizes profit $y_2 + p_1 y_1$ subject to $y_2 = g(y_1)$, giving that $g' = -p_1$. Collecting the two expressions, we get that

$$\frac{u_1'}{u_2'} + g' + t = 0$$

or $u_1' + u_2' g' + t u_2' = 0$. If the tax is set as $t = \frac{u_3'}{u_2'}$, we get the optimality criterion (1).

If we introduce an Arrow commodity with number 3 interpreted as "externality to the consumer from the producer", we get that the consumer finds the optimal bundle (x_1, x_2, x_3) by maximizing utility under budget constraint (we keep the normalization rule $p_2 = 1$)

$$p_1 x_1 + x_2 + p_3 x_3 = 0$$

with first order conditions $\frac{u_1'}{u_2'} = p_1$, $\frac{u_3'}{u_2'} = p_3$, and the producer maximizes the profit $y_2 + p_3 y_3 + p_1 y_1$ subject to the production functions $y_2 = g(y_1)$, $y_3 = -y_1$, to give the first order condition

$$g' = -p_1 - p_3.$$

Collecting, we get that

$$g' = -\frac{u_1'}{u_2'} - \frac{u_3'}{u_2'}$$

and multiplying by u_2' we get the equation (1).

(2) Assume that in the set $L = \{1, \ldots, l\}$ of commodities, there is a subset L_s of commodities which are related to healthcare, say treatment of specific diseases. If the Samaritan principle is at work, each individual i is concerned with the amount x_{jh} of consumption of healthcare of type h by consumer j. If preferences are described by utilities, the utility function of consumer i will have the form

$$u_i(x_i, (x_{jh})_{i=1\ h\in L_s}^m).$$

This is clearly an externality in consumption, and Pareto optimal allocation cannot be expected to be obtainable as market equilibria, intuitively since money transfers alone does not secure that other consumers buy the amount of healthcare that we want them to consume.

Formally, this could be solved by introducing Arrow commodities ξ_{ijh} for the externality to consumer i induced by consumer j's consumption of commodity $h \in L_s$, to be sold by consumer j who produces this commodity using commodity h as input, with associated prices q_{ijh}.

In the artificial economy constructed in this way, each consumer j is also a producer, using commodity $h \in L_s$ as input in a production process which gives the input back as all the Arrow commodities ξ_{ijh} (where $\xi_{jjh} = x_{jh}$ is the ordinary consumption by j of commodity h). Allocations in this economy can be identified with allocations in the original economy, and Pareto optimal allocations in one economy is Pareto optimal in the other as well. Since there are no externalities, Pareto optimal allocations can be obtained as market equilibria in the artificial economy. Clearly, the Arrow commodities are of a form which cannot easily be implemented practically. In simple cases where externalities influence on all consumers in roughly the same way and do not depend on the individual involved, the Arrow commodities can be interpreted as subsidized or free healthcare.

(3) We normalize prices so that $p_1 = 1$. Let the personalized prices for commodity 2 be q_1 and q_2, respectively, and the producer price be $q = q_1 + q_2$. By constant returns to scale, profits are 0, so that $qy_2 - y_1 = 2qy_1 - y_1 = 0$ and $q = \frac{1}{2}$.

The demand for commodity 2 by consumer 1 having a Cobb-Douglas utility is $\frac{1}{3}\frac{6}{q_1}$, and since this should equal the amount produced, which is $2y_1$, we get that

$$y_1 q_1 = 1.$$

For consumer 2, we have that demand for the two commodities has equal size, and from the budget equation we get that $(1 + q_2)2y_1 = 8$ or

$$y_1 + y_1 q_2 = 4.$$

Adding the two equations and using that $q_1 + q_2 = q = \frac{1}{2}$, we obtain that $\frac{3}{2}y_1 = 5$ or $y_1 = \frac{10}{3}$. Using the previous equations we find that $q_1 = \frac{3}{10}$ and $q_2 = \frac{1}{5}$. The first consumer demands $\frac{2}{3}6 = 4$ units of commodity 1, and consumer 2 then gets $14 - \left(4 + \frac{10}{3}\right) = \frac{20}{3}$ units, which (as it should be) is identical to the consumption of commodity 2, which is $2 \cdot \frac{10}{3} = \frac{20}{3}$.

Summing up, we have found the Lindahl equilibrium

$$\left(\left(4, \frac{20}{3}\right), \left(\frac{20}{3}, \frac{20}{3}\right), \left(-\frac{10}{3}, \frac{20}{3}\right), \left(1, \left(\frac{3}{10}, \frac{1}{5}\right)\right)\right).$$

(4) If $x_2(c) = \sum_{i=1}^m c_i$ is produced, then the input requirement is $\frac{1}{K}\sum_{i=1}^m c_i$, which is equal to the sum of payments $\tau_i(c)$ for $i = 1, \dots, m$,

$$\sum_{i=1}^m \tau_i(c) = m\frac{1}{m}\frac{1}{K}x_2(c) + \sum_{i=1}^m [c_{i+2} - c_{i+1}]x_2(c) = \frac{1}{K}\sum_{i=1}^m c_i$$

since the second sum vanishes.

We have that

$$\frac{\partial \tau_i(c)}{\partial c_i} = \frac{1}{m}\frac{1}{K} + c_{i+2} - c_{i+1},$$

and given that individual i has final utility $u_i(\omega_i - \tau_i(c), \sum_{i=1}^m c_i)$ we get from the Nash equilibrium property that

$$\frac{\partial u_i}{\partial c_i} = -u'_{i1}\left[\frac{1}{m}\frac{1}{K} + c_{i+2} - c_{i+1}\right] + u'_{i2} = 0,$$

so that

$$\frac{u'_{i2}}{u'_{i1}} = \left[\frac{1}{m}\frac{1}{K} + c_{i+2} - c_{i+1}\right],$$

and summation over i gives

$$\sum_{i=1}^m \frac{u'_{i2}}{u'_{i1}} = \frac{1}{K},$$

which is the condition for Pareto optimality.

(5) For the local game at any instant, if individual i gets the share $\theta_i > 0$ of a possible surplus, then the expected utility gain at the mesage Δx_i is

$$\Delta u_i(\Delta x_1, \dots, \Delta x_m) = \begin{cases} -u'_{i1}\left(\Delta x_i - \theta_i\left(\sum_{j=1}^m \Delta x_i - C'\Delta y\right)\right) + u'_{i2}\Delta y, & \sum_{j=1}^m \Delta x_j \geq C'(\Delta y) \\ 0 & \sum_{j=1}^m \Delta x_j < C'(\Delta y). \end{cases}$$

Let Δx_i^0 be such that

$$-u'_{i1}\Delta x_i^0 + u'_{i2}\Delta y = 0,\ \text{or}\ \Delta x_i^0 = \frac{u'_{i2}}{u'_{i1}}\Delta y,$$

so that Δx^0 reflects the true willingness to pay for the increased provision. For any array $(\Delta x_j)_{j=1}^m$, if $\Delta x_i^0 + \sum_{j\neq i}\Delta x_j < C'\Delta y$, then sending a message $\Delta x_i < \Delta x_i^0$ will change nothing, and sending $\Delta x_i > \Delta x_i^0$ may result in increased production but at a cost which is too high for individual i. If $\Delta x_i^0 + \sum_{j\neq i}\Delta x_j > C'\Delta y$, then $\Delta x_i < \Delta x_i^0$ may lead to production, in other cases (such as $\Delta x_j = \Delta x_j^0$ for all j, it may result in no production which is worse. Finally, $\Delta x_i > \Delta x_i^0$ would not increase payoff since the individual only gets a share θ_i of the redundant payment. We conclude that true willingness to pay is maximin since it is best in worst case of the other individuals' messages.

Since at each instant, the change in consumption and production given that all choose their true willingness to pay satisfies

$$\Delta u_i(\Delta x_1^0, \ldots, \Delta u_m^0) \geq 0, i = 1, \ldots, m,$$

we get that the process continues until no increase in production of the public good will increase utility, so that

$$C'\Delta y = \sum_{i=1}^m \Delta x_i^0 = \sum_{i=1}^m \frac{u'_{i2}}{u'_{i1}}\Delta y$$

or

$$\sum_{i=1}^m \frac{u'_{i2}}{u'_{i1}}$$

showing that the resulting allocation is Pareto optimal.

(6) We normalize prices so that $p_1 = 1$. Then the price system which is normal to the production set at (y_1, y_2) has the form $\left(1, \frac{1}{2y_2}\right)$, and the profit at $(y_1, y_2) = (y_1, y_1^2)$ is $\frac{y_1}{2} - y_1 = -\frac{y_1}{2}$, which is distributed to consumers with $-\frac{y_1}{6}$ to each.

The demand for commodity 1 from consumers 1,2 and 3 is

$$\frac{1}{2}\left[1 + \frac{1}{y_1} - \frac{1}{6}y_1\right] + \frac{1}{3}\left[1 + \frac{1}{y_1} - \frac{1}{6}y_1\right] + \frac{1}{4}\left[2 + \frac{1}{y_1} - \frac{1}{6}y_1\right]$$

which should equal the net supply $4 - y_1$, giving the second degree equation

$$\frac{85}{72}y_1 - \frac{19}{12} + \frac{29}{12} = 0$$

with positive solution $y_1 = 2.7785$, giving the production $(y_1, y_2) = (-2.7785, 7.72)$, price system $(1, 0.1780)$, and the equilibrium is

$$\begin{aligned}&(x_1, x_2, x_3, y, p)\\ &\quad = ((0.4484, 2.4918), (0.2989, 3.3224), (0.4742, 7.9055), (-2.7785, 7.72), (1, 0.1780).\end{aligned}$$