Exercises to Chapter 11

(1) For simplicity we assume that we are dealing with an exchange economy. At the equilibrium (x, p), net savings (positive or negative) of consumer i at date t is given by

$$s_{it} = p_t \cdot \omega_{it} - p_t \cdot x_{it},$$

i.e. as the difference between income derived from selling endowments related this date and consumption expenditure at that date. In the case that net savings are negative they may be considered as loan offered by consumer i.

Balance over time of net savings follows simply from

$$\sum_{t=0}^{T} s_{it} = \sum_{t=0}^{T} p_t \cdot (\omega_{it} - x_{it}) = p \cdot (\omega_i - x_i) = 0$$

holding for each consumer i in equilibrium.

(If we consider an economy with production, then the definition of the savings of a consumer depends on how profit shares are treated. Assuming that profits are distributed only at T, we get that sum of (typically negative) net savings will be equal to total profit. Alternatively profit may be distributed each year, but then profit may be negative in some years, corresponding to additional financing by the owners of the firm.)

In the case of negative net saving (loan), the amount $-s_{i0}$ can be seen as a loan l_{i0} obtained by i (bond issued) to be repaid in the next period. If $s_{i0} > 0$, then l_{i0} is a credit offered (bonds bought) The repayment of l_{i0} at $t = 1$ indices a new loan l_{i1} to cover net savings and repayment, $l_{i1} = -s_{i1} - l_{i0}$, and in general

$$l_{i,t+1} = -s_{i,t+1} - l_{it},$$

so that the outstanding bonds of consumer i are exactly equal to the (negative) net savings.

If the ratio λ_t of bonds issued are not repaid, then the budget inequality of consumers is

$$p \cdot x_i \leq p \cdot \omega_i + \sum_{t=0}^{T-1} \lambda_t \max\{p_t \cdot (\omega_{it} - x_{it}), 0\} - \sum_{t=1}^{T} \tau_t p_t \cdot \omega_{it},$$

with τ_t the rate for taxing endowments at dates $t = 1, \dots, T$. In an equilibrium (x, p), consumers choose their bundles so that it is maximal for the preference relation on the budget set, and the tax rates are set so that where

$$\tau_t = \frac{\sum_{j=1}^m \lambda_{t-1} \max\{p_{t-1} \cdot (\omega_{j,t-1} - x_{j,t-1}), 0\}}{\sum_{j=1}^m p_t \cdot \omega_{jt}}, t = 1, \dots, T.$$

Since the equilibrium is a market equilibrium with incomes determined by endowments modified by transfers, the allocation will be Pareto optimal.

(2) Considering $Y_t = p_t \cdot y_t$ as the cash flow at time t, measured in discounted prices, it may be considered as a cash flow which will occur after a waiting time of t periods. With this interpretation, MacD is a weighted average of the waiting times for the total cash flow $\sum_{t=1}^T Y_t$.

If $\bar{p}_{Kt} = p_{K0}(1+i)^{-t}$ for all i, we have that $p_t \cdot y_t = \bar{p}_t \cdot y_t (1+i)^{-1}$, all t, where $\bar{p}_t$ is the vector of non-discounted prices at t. The net present value of $(Y_t)_{t=1}^T$ is then

$$V = \sum_{t=1}^T \bar{Y}_t (1+i)^{-t},$$

with $\bar{Y}_t = \bar{p}_t \cdot y_t$, and its derivative with respect to $1 + i$ is

$$\frac{\mathrm{d}V}{\mathrm{d}(1+i)} = -\frac{\sum_{t=1}^T t\bar{Y}_t(1+i)^{-t}}{1+i},$$

and the elasticity of NPV wrt. $1 + i$ is

$$-\frac{\frac{\mathrm{d}V}{\mathrm{d}(1+i)}}{\frac{V}{1+i}} = -\frac{\sum_{t=1}^T t\bar{Y}_t(1+i)^{-t}}{\sum_{t=1}^T \bar{Y}_t(1+i)^{-t}},$$

which is MacD. Thus, the Macaulay duration can be seen as a sensitivity measure of the NPV with respect to the interest rate.

(3) Since endowments are bounded over time in this economy, we may take as prices all sequences $(p_t)_{t=0}^\infty$ such that $\sum_{t=0}^\infty p_t < \infty$. A Walras equilibrium is then an array (c_1, c_2, p) with $c_i = (c_i^t)_{t=0}^\infty$, $i = 1, 2$, and $p = (p_t)_{t=0}^\infty$, such that

(i) for $i = 1, 2$, c_i maximizes $\sum_{i=0}^\infty \beta^t u(x_i^t)$ over all $x = (x_i^t)_{t=0}^\infty$ such that $\sum_{t=x0}^\infty p_t(x_i^t - \omega_i^t) = 0$,

(ii) for $t = 0, 1, \dots$, $c_1^t + c_2^t = \omega_1^t + \omega_2^t$.

In the given case, everything repeats itself after three periods, and since utilities are concave, optimal consumption programs will be such that consumption $c_i^t = c_i^0$ is constant over time. Since marginal rate of substitution must equal price ratios, we have for every pair $t, \tau \in \mathbb{Z}_+$ that

$$\frac{p_t}{p_\tau} = \frac{\beta^t u'(c_i^0)}{\beta^\tau u'(c^0)} = \frac{\beta^t}{\beta^\tau},$$

and choosing the good at date 0 as numeraire we get that $p = (\beta^t)_{t=0}^\infty$. The consumption bundle c_1^0 is now found using the three-period budget equation

$$c_1^0 + \beta c_1^0 + \beta^2 c_1^0 = 1$$

with solution $c_1^0 = \dfrac{1}{1+\beta+\beta^2}$, and similarly we find that $c_2^0 = \frac{\beta+\beta^2}{1+\beta+\beta^2}$.

The value of the bond yielding a payment of 0.1 at each period will be $0.1\beta + 0.1\beta^2 + \cdots = \dfrac{\beta}{1-\beta}$.

(4) Normalizing prices so that $p_1 = 1$, we have that the income under perfect markets given the price expectations would be

$$w_1 = 4 + p_2 + 0.4 + 5p_2,\ w_2 = 2 + 2p_2 + 2.2 + 3.3p_2.$$

If both consumers would buy commodity 2 in order to transfer to next period, then demand would be as with perfect markets, and demand = supply for commodity 1 means that

$$\frac{1}{4}(4.4 + 6p_2) + \frac{1}{7}(4.2 + 5.3p_2) = 6$$

or $p_2 = 1.905$. At this price, demand exceeds supply, showing that utility maximization under a single budget constraint would mean that consumers buy too much compared to what is available. Since only saving, not borrowing, si possible, the economy splits in two periods, so that demand is derived using only income available in the current period. Then the equilibrium condition for commodity 1 becomes

$$\frac{1}{2}(4 + p_2) + \frac{2}{3}(2 + 2p_2) = 6$$

with solution $p_2 = 2.857$. It is easily seen that at these prices, no consumer wants to save, so we have found a temporary equilibrium.

(5) We consider the economy over time, $t = 0, 1, \ldots$. There is one consumption good in each period, the capital good 'land' can be used as collateral and for production. There are two types of agents:

(i) *entrepreneur-borrowers* own the land but have no endowment of consumption good

(ii) *lenders* with endowments of consumption good

One unit consumption good plus k units of land to yield y units of t consumption good in the next period.

At date t, the entrepreneur can obtain $q_{t+1}(1+r)^{-1}$ of the consumption good as a loan using land as collateral. With this amount of the good, the entrepreneur needs $kq_{t+1}(1+r)^{-1}$ units of land, and the The remaining $1 - kq_{t+1}(1+r)^{-1}$ is rented out at a rate h_t, giving income $h_t[1 - kq_{t+1}(1+r)^{-1}]$. We assume that rent is determined by $h_t = b - al_t$.

The relation between land price in t and $t+1$ is then given by

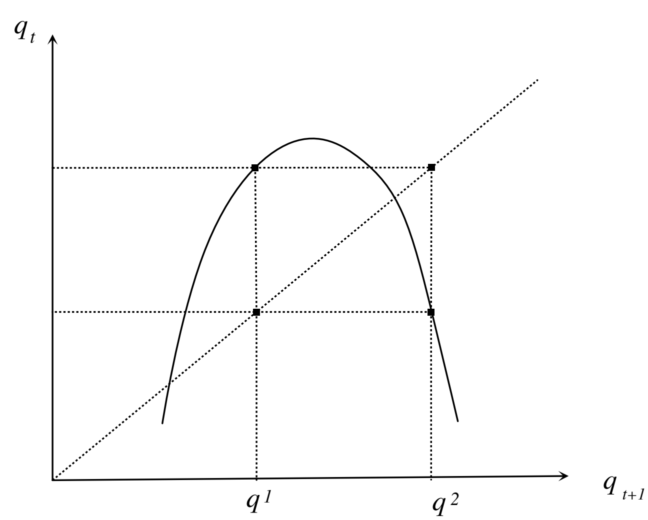
$$q_{t+1} + (y - (1+r))\frac{q_{t+1}}{1+r} + h_t\left(1 - \frac{kq_{t+1}}{1+r}\right) = q_t(1+r).$$

(keeping the land, producing and renting out what remains, should be just as good as selling now).

Inserting h_t, this gives q_t is a second-degree polynomial in q_{t+1},

$$q_t = \phi(q_{t+1}).$$

The graph of this polynomial will have this form



It is seen that q_t exhibits 2-cycles.