Exercises to Chapter 12

(1) For the generation born at t, if prices are (p_1^t, p_2^t) in the first year and optimal saving of consumer i is s_i^t, then demand of consumer 1 and 2 for commodity 1 are

$$\frac{1}{2}\,\frac{3(p_1^t + p_2^t) - s_1^t}{p_1^t} + \frac{2}{3}\,\frac{3(p_1^t + p_2^t) - s_2^t}{p_1^t},$$

and the demand of the old generation is

$$\frac{1}{2}\,\frac{p_1^t + p_2^t + s_1^{t-1}}{p_1^t} + \frac{2}{3}\,\frac{p_1^t + p_2^t + s_2^{t-1}}{p_1^t}$$

so that total demand of commodity 1 is

$$\frac{14}{3}\,\frac{p_1^t + p_2^t}{p_1^t} + \frac{1}{2p_1^t}(s_1^{t-1} - s_1^t) + \frac{2}{3p_1^t}(s_1^{t-1} - s_1^t)$$

which should equal the total endowment of commodity 1, namely 8, and similarly for commodity 2,

$$\frac{10}{3}\,\frac{p_1^t + p_2^t}{p_1^t} + \frac{1}{2p_1^t}(s_1^{t-1} - s_1^t) + \frac{1}{3p_1^t}(s_1^{t-1} - s_1^t) = 8.$$

In a hypothetical steady state where $s_i^{t-1} = s_i^t$ for $i = 1, 2$, we would have that relative prices (assuming $\hat{p}_1^t + \hat{p}_2^t = 1$) will be $\hat{p}_1^t = \frac{7}{12}, \hat{p}_2^t = \frac{5}{12}$. The absolute prices could then be found from $p_h^t = 1.05 p_h^{t-1}$ since marginal utility from transfering income between periods should correspond to its cost.

Unfortunately, optimal savings are not time-independent since the initial generation has no saving, so the sequence of optimal savings will only approach a fixed level over time, after which the prices will be as above.

(2) [Typo in the definition of price support, should be: A price system $(p_t)_{t=1}^{\infty}$ with $p_t > 0$, all t, supports the program $(x_t, y_t, c_t)_{t=1}^{\infty}$ if $p_{t+1} f_t(x_t) - p_t x_t \geq p_{t+1} f_t(x_t') - p_t x_t'$ for all x_t', all t.]

For each t, let $\hat{q}_t$ be a supergradient of the concave function f_t at x_t, i.e. $\hat{p}_t$ satisfies

$$f_t(x') \geq f_t(x_t') + \hat{q}_t(x_t')(x_t' - x_t),$$

and define the price system $(p_t)_{t=1}^{\infty}$ inductively such that

$$\hat{q}_1 = \frac{1}{\hat{p}_1},\ \hat{q}_t = \frac{p_{t-1}}{p_t}, t \geq 1.$$

Then $(p_t)_{t=1}^{\infty}$ supports the program (x_t, y_t, c_t) (efficiency of the program is not needed).

A simple example of a price-supported program which is not efficient is the following: Let the production function be time-independent,

$$f(x) = 2\ln(x+1).$$

Then the steady-state program with

$$(x, y, c) = \left(\frac{1}{2}, 2\ln\left(\frac{3}{2}\right), 2\ln\left(\frac{3}{2}\right) - \frac{1}{2}\right) = (0.5, 0.81, 0.31)$$

is dominated by the steady-state program $(1, \ln 2, \ln 2 - 1) = (1, 1.39, 0.39)$ (which is actually the golden rule program with $f'(1) = 1$).

(3) Given a program $(x_t^0, y_t^0, c_t^0)_{t=1}^{\infty}$ [superscript '0' missing in text], the family of sets

$$S_t = \{(x - x_t^0, y_{t+1}^0 - y) \mid y \leq f_t(x)\},\ t = 0, 1, \ldots$$

is indeed a reduced model: Each S_t is a subset of $\mathbb{R}^2$, $(0,0) = (x_t^0 - x_t^0, y_{t+1}^0 - f_t(x_t^0))$ belongs to S_t, and finally, if $(x - x_t^0, y_{t+1}^0 - y)$ and $(r, s) \in \mathbb{R}_+^2$, then

$$y - s \leq y \leq f_t(x) \leq f_t(x + r),$$

so that $((x - x_t^0) + r, (y_{t+1}^0 - y) + s) = ((x + r) - x_t^0, y_{t+1}^0 - (y - s))$ belongs to S_t.

If $(\varepsilon_t)_{t=0}^{\infty}$ is an improving sequence for $\mathcal{S} = (S_t)_{t=1}^{\infty}$, define $x_t' = x_t^0 - \varepsilon_t$ for $t \in \mathbb{N}$. Since $(-\varepsilon_t, \varepsilon_{t+1}) \in S_t$, we have that

$$x_{t+1}^0 - x_{t+1}' = y_{t+1}^0 - y_{t+1}'$$

for some $y_{t+1}' \leq f_t(x_{t+1}'$, all t. Letting $\tau = \min\{t \mid \varepsilon_t \neq 0\}$. If $\varepsilon_\tau < 0$, then $\tau > 1$ and $(0, \varepsilon_t) \in S_{\tau-1}$, so that there is $y_\tau' > y_\tau^0$ for which $y_\tau' \leq f_{\tau-1}(x_{\tau-1}') = f_{\tau-1}(x_{\tau-1}^0)0y_\tau^0$, a contradiction, so $\varepsilon\tau > 0$.

Define $(y_t'')_{t\in\mathbb{N}}$ by $y_t'' = y_t^0$ for $t \leq \tau$, and define $(c_t')_{t\in\mathbb{N}}$ by $c_t' = y_t'' - x_t'$ for $t \in \mathbb{N}$. Then $c_t' = c_t^0$ for $t \neq \tau$ and $c_\tau' > c_\tau^0$, contradicting efficiency of $(x_t^0, y_t^0, c_t^0)_{t\in\mathbb{N}}$. Thus, there cannot be an improving sequence for $\mathcal{S}$.

Given that efficiency in the production model is equivalent to absence of improving sequences in the reduced model, the result of Theorem 12.3 can be extended immediately, so that the program $(x_t^0, y_t^0, c_t^0)_{t=1}^{\infty}$ with price support $(p_t)_{t=1}^{\infty}$ is efficient if $\liminf_{t\to\infty} p_t x_t^0 = 0$.

(4) We consider an overlapping generations model with two countries A, B, in each country the generations are identical over time and consist of a single consumer living for two consecutive periods of time, and there is only one good. In country A the individual has endowment $(1,1)$, and at the point $x_A = \left(\frac{1}{2}, \frac{5}{4}\right)$ the preferred set is

$$P_A(x_A) = \left\{x \in \mathbb{R}^2_+ \,\middle|\, (3,2)\cdot x > (3,2)\cdot x_A = 4\right\}.$$

The consumer of country B has also endowment $(1,1)$, and at the point $x_B = \left(\frac{3}{2}, \frac{3}{4}\right)$ the preferred set is $P_B(x_B) = \{x \in \mathbb{R}^2_+ \,|\, (3,2)\cdot x > (3,2)\cdot x_B\}$.

The allocation-price pair (x_A, x_B, p) with $p = (3,2)$ is a Walras equilibrium, whereby the consumer of generation t in A has the net trade $x_A - (1,1)$ with the consumer of generation t in B. It can however be improved: Suppose that from some given date t^0 the young consumer in B gives a small amount Δ of the good to the old consumer (who will then be better off). To compensate, the consumer gets the same amount back as old. This redistribution between generations in B does not interfere with the bundle obtained by the consumer in A, so the new allocation is feasible, and for small enough Δ it is preferred by all in B (after t_0), so that the Walras equilibrium cannot be Pareto optimal.

(5) Let (x, p) be a Walras equilibrium in the OLG economy (cf. p.314). Here we can rewrite the budget constraint (2) as

$$\sum_{h=1}^{l-1} p_h^t x_{ih}^t + p_l^t \mu_i^t + \sum_{h=1}^{l-1} p_h^{t+1} x_{ih}^{t+1} + p_l^{t+1}\mu_i^{t+1} \le \sum_{h=1}^{l-1} p_h^t \omega_{ih}^t + \sum_{h=1}^{l-1} p_h^{t+1}\omega_{ih}^{t+1}$$

where μ_i^t is the change (possibly negative) in holding of commodity l at t. Assuming that commodity l in the is irrelevant for preferences of the consumers, we have that $\mu_i^{t+1} = -\mu_i^t$.

For a change in the total endowment of commodity l, it matters for the consequences whether it is anticipated or not. If anticipated, the equilibrium will change since the old generation has a larger endowment at t. If not anticipated, the previous equilibrium will continue from $t+1$ and onwards, only the prices of commodity l have changed corresponding to the larger stock of the commodity.

(6) [Incorrect sign in the assumption on derivatives, derivation w.r.t. $r = p^{t+1}/p^t$ means that $\mathrm{D}\xi^0$ typically will be positive due to substitution] In order to apply Thm.12.5, one needs to assure that the conditions (i)-(iv) (adapted to the context) in the theorem are satisfied. In particular, if r is increased from 1, then ξ^0 will typically increase to some $\bar{x}^0 > \xi^0(1)$, and if r is decreased from 1, then ξ^1 will increase to some $\hat{x}^1 > \xi^1(1)$, so that

$$\bar{x}^0 + \hat{x}^1 > \xi^0(1) + \xi^1(1) > \omega_1 + \omega_2,$$

and (iv) cannot hold. The assumption for this to work $D\xi^0(1) - D\xi^1(1) > 0$ or $D\xi^0(1) > D\xi^1(1)$.