Exercises to Chapter 2

(1) The economy $\mathcal{E}$ satisfies the conditions for existence of a Walras equilbrium, and from the form of the utility functions it is seen that the equilibrium prices must be positive. We may therefore normalize prices by the setting $p_1 = 1$. Also, we may restrict attention to one of the commodities, say commodity 1, since equality of supply and demand for this commodity will entail equality also for the other commodity.

The consumers have demand functions $\frac{3}{4}I_1$, $\frac{1}{2}I_2$ and $\frac{1}{4}I_3$ (for commodity 1), where I_i is the income of consumer $i = 1, 2, 3$, giving a total demand of

$$\frac{3}{4}I_1 + \frac{1}{2}I_2 + \frac{1}{4}I_3. \tag{1}$$

To find the incomes we need expressions for the profit the two firms: Profit maximization at the price vector $(1, p_2)$ yields the optimal production $\left(\frac{1}{2p_2}, \frac{1}{4p_2^2}\right)$ and profit $\frac{1}{4p_2}$ in the first firm and $\left(\frac{1}{(4p_2)^{1/3}}, \frac{1}{(4p_2)^{4/3}}\right)$ and profit $\frac{3}{(4p_2)^{1/3}}$.

We now get that

$$\begin{aligned} I_1 &= \frac{1}{10} + \frac{9}{10}p_2 + \frac{1}{3}\Pi \\ I_1 &= \frac{2}{10} + \frac{8}{10}p_2 + \frac{1}{3}\Pi \\ I_1 &= \frac{3}{10} + \frac{7}{10}p_2 + \frac{1}{3}\Pi \end{aligned}$$

with $\Pi = \frac{1}{4p_2} + \frac{3}{(4p_2)^{1/3}}$

Inserting in (1) and setting it equal to supply, which is

$$\frac{6}{10} + \frac{1}{2p_2} + \frac{1}{(4p_2)^{1/3}},$$

one gets an equation in p_2,

$$\frac{5}{4}p_2 - \frac{3}{8}\frac{1}{p_2} - \frac{5}{8}\frac{1}{(4p_2)^{1/3}} - \frac{14}{60} = 0,$$

which can be solved to give $p_2 = 0.9268$. Inserting in the expressions for individual demand and supply one gets the allocation.

(2) Suppose that y_j^0 maximizes $p \cdot y_j$ on Y_j for $j = 1, 2$, and let $y_j^* \in Y_1 + Y_2$ be arbitrary. Then $y^* = y_1^* + y_2^*$ for some $y_h^* \in Y_j$, $j = 1, 2$, and since $p \cdot y_j^* \le p \cdot y_j^0$, $j = 1, 2$, we get that

$$p \cdot y^* = p \cdot (y_1^* + y_2^*) \le p \cdot (y_1^0 + y_2^0),$$

showing that $y_1^0 + y_2^0$ maximizes $p \cdot y$ on $Y_1 + Y_2$.

Conversely, suppose that y^0 maximizes $p \cdot y$ on $Y_1 + Y_2$, and write $y^0 = y_1^0 + y_2^0$. If for some j, say $j = 1$, there is $y_1^* \in Y_1$ with $p \cdot y_1^* > p \cdot y_1^0$, then we would have that

$$p \cdot (y_1^* + y_2^0) \ge p \cdot (y_1^0 + y_2^0) = p \cdot y^0,$$

a contradiction, and we conclude that y_j^0 maximizes $p \cdot y_j$ on Y_j for $j = 1, 2$.

(3) The matrix A is productive by Lemma 2.4, since (a)

$$I - A = \begin{pmatrix} 0,85 & -0,25 \\ -0,70 & 0,95 \end{pmatrix} \text{ is regular, } \det(I - A) = 0,6325 > 0,$$

and the its inverse,

$$(I - A)^{-1} = \frac{1}{0,6325} \begin{pmatrix} 0,95 & -0,70 \\ 0,25 & 0,85 \end{pmatrix},$$

has only positive elements.

The matrix A has a double eigenvalue μ, found by solving the equation

$$\det \begin{pmatrix} 0,85\mu & -0,25 \\ -0,70 & 0,95\mu \end{pmatrix} = 0,$$

which gives the value $\mu = 0,4472$.

(4) We look for an activity vector (x_1, x_2, x_3), a growth rate α and a price vector (p_1, p_2, p_3) such that

$$xB \ge \alpha xA,\ a(B - \alpha A) \cdot p = 0,\ p \cdot e_i(B - \alpha A) \le 0,\ i = 1, 2, 3,$$

where e_i is the ith unit vector. The growth rate α should be determined as

$$\max_{x \in \Delta} \min_{i=1,2,3} \frac{xB_i}{xA_i}$$

and it should be minimal such that $p \cdot e_i B \le \alpha p \cdot e_i A$. An approximate solution can be found as $\alpha = 1.64$, $x = (0.31, 0, 15, 0.53)$, with $p = (0.9, 0, 49, 0)$.

(5) Suppose that P is decomposable, so that there is $\emptyset \neq J \subset \{1, \ldots, n\}$ such that $p_{ij} = 0$ when $i \notin J$, $j \in J$. Choose $i^0 \notin J$ and $j^0 \in J$ arbitrarily. If there was a path from i^0 to j^0 with nonzero probabilities, then there would be a first time that the path entered J, which means that it would have to come from $\{1, \ldots, n\} \backslash J$, which is impossible since P is decomposable. We conclude that P cannot be irreducible.

Conversely, suppose that P ia not irreducible, so that there are i^*, j^* which cannot be connected by a path associated with nonzero probabilities. Define $I^* = \{i \mid i \text{ is connected to } i^0\}$, and let $J^* = \{1, \ldots, n\} \backslash I^*$. Then J^* is nonempty since it contains j^*. If $i \in I^*$ and $j \in J^*$, then $p_{ij} = 0$ by the definition of I^*. It follows that P is decomposable.

Since P is a row stochastic matrix (that is $\sum_{j=1}^n p_{ij} = 1$ for each i), it has the eigenvector $(1, \ldots, 1)$, and by Perron-Frobenius $\lambda(P) = 1$. It follows that also the transpose P^t of P has eigenvalue 1, so that there is a unique (except for scalar multiple) positive vector x^* with $P^t x^* = x^*$. Normalizing x^* we get that P is ergodic with limiting distribution x^*.

(6) *Rybczinski's theorem* [the name is misspelled in the text]: Given the standard assumptions of the Heckscher-Ohlin model, the factor proportions in the two industries are determined uniquely by the relative prices of the finished goods, which is kept constant. If total endowment $(\overline{m}^0, \overline{\ell}^0)$ in a country is changed to, say, $(\overline{m}^1, \overline{\ell}^0)$ with $\overline{m}^1 > \overline{m}^0$, and the balancing equation

$$(m_1, \ell_1) + (m_2, \ell_2) = (\overline{m}^1, \overline{\ell}^0)$$

must be satisfied with unchanged ratios between m and ℓ in the industries, output must be increased in the industry with the big m_j/ℓ_j, that is in the industry which uses m intensively. For increase in ℓ the reasoning is similar.

The Stolper-Samuelson theorem: Suppose without loss of generality that the first industry is relatively m-intensive (so that the contract curve is below the diagonal in the Edgeworth box). Since the factor proportion in each industry is monotonically increasing along the contract curve, an increase in output in the first industry means that the factor proportion increases, and the relative price of m increases. Given that the amount of factors used in the country is unchanged, the incomes of the owners of m increase and those of the owners of ℓ decrease.