Exercises to Chapter 3

(1) In a Pareto optimal allocation the marginal rates of substitution in production and in consumption must be the same,

$$\frac{u_2'}{u_1'} = \frac{1}{2}\frac{x_1}{x_2} = -\frac{1}{\sqrt{-2y_1}} = \frac{\mathrm{d}y_2}{\mathrm{d}y_1},$$

and inserting $x_1 = 20 + y_1$ and $x_2 = \sqrt{-2y_1}$ one gets that $y_1 = -18$, so that the unique Pareto optimal allocation is given by $(x, y) = ((2, 6), (-18, 6))$. The price at which this allocation is obtained as market equilibrium is

$$\frac{u_2'}{u_1'} = \frac{1}{2}\frac{x_1}{x_2} = \frac{1}{6}.$$

(2) [For this result, we need to assume that preferences P_i are described by continuous utility functions u_i.] Assume that $(x_1, \ldots, x_m)$ is maximally egalitarian equivalent with some reference bundle $\lambda\overline{x}$, so that $u_i(x_i) = u_i(\lambda\overline{x})$ for all i and λ is maximal with this property. If the feasible allocation $x' = (x_1', \ldots, z_m')$ satisfies $u_i(x_i') \geq u_i(x_i)$ for all i and $u_j(x_j') > u_j(x_j)$ for at least some j, then after some small redistribution of commodities we get a feasible allocation $x'' = (x_1'', \ldots, x_m'')$ satisfying

$$u_i(x_i'') > u_i(\lambda\overline{x})$$

for all i. Turning again to redistribution if necessary, we may assume that $u_i'(x_i'') = u_i(\lambda''\overline{x})$ for all i, with $\lambda'' > \lambda$, a contradiction since λ was maximal with this property.

(3) For each i, let X_i^0 be a the consumption set of i truncated so as to contain $\{x_i \in X_i \mid p \cdot x_i = p \cdot \omega_i\}$. X_i^0 is convex and compact. For each i, define $w_i : X_i^0 \times \Delta \to \mathbb{R}$ by

$$w_i(x_i, q) = q \cdot \omega_i - \lambda[p \cdot (x_i - \omega_i)],$$

where $\lambda > 0$ is chosen so small that X_i^0 contains elements x_i' with $q \cdot x_i' < w_i(x_i, q)$ for all q Now, let $\gamma_i : X_i^0 \times \Delta \rightrightarrows X_i^0$ be defined by

$$\gamma_i(x_i, q) = \left\{x_i' \in X_i^0 \,\middle|\, q \cdot x_i' < w_i(x_i, q)\right\},$$

and define $\phi_i : \gamma : X_i^0 \rightrightarrows X_i^0$ by

$$\phi_i(x_i, q) = \begin{cases} \gamma_i(x_i, q) & x_i \in \operatorname{cl}\gamma_i(x_i, q) \\ \gamma_i(x_i, q) \cap P_i(x_i) & \text{otherwise.} \end{cases}$$

Finally, define $\phi_0 : \prod_{i=1}^m X_i^0 \rightrightarrows \Delta$ by

$$\phi_0(x_1, \dots, x_m) = \left\{ q \in \Delta \,\middle|\, q \cdot \sum_{i=1}^m (x_i - \omega_i) > 0 \right\}$$

The correspondences $\phi_0, \phi_1, \dots, \phi_m$ have convex, possibly empty values and open graph. Reasoning as in the proof of Theorem 1.1, one finds that there is $(x_1^0, \dots, x_m^0, q^0) \in \prod_{i=1}^m X_i^0 \times \Delta$ such that

$$\phi_i(x_i^0, q^0) = \emptyset, i = 1, \dots, m, \phi_0(x_1^0, \dots, x_m^0) = \emptyset.$$

Since $\phi_i(x_i^0, q^0) = \emptyset$, we must have that $p \cdot (x_i - \omega_i) = 0$ and consequently $q \cdot x_i \leq q \cdot \omega_i$ for each i, and from $\phi_0(x_1^0, \dots, x_m^0) = \emptyset$ we get that $\sum_{i=1}^m x_i^0 = \sum_{i=1}^m \omega_i$.

We thus have that x^0 is budget constrained and feasible. If another allocation x' satisfies $x_i' \in \operatorname{cl} P_i(x_i^0)$ for all i and $x_{i'}' \in P_{i'}(x_{i'}^0)$ for some i', then $\sum_{i=1}^m q^0 \cdot x_i' > \sum_{i=1}^m x_i^0 = \sum_{i=1}^m \omega_i$, so x' cannot be feasible. We conclude that x^0 is Pareto optimal as well.

(4) For the allocation part of the intervention, we may use the approach leading to the expression (6) on p.99: Assuming that social welfare of allocations of commodities and health $(x_1, \dots, x_m, h_1, \dots, h_m)$ can be measured by social welfare function S, then the change in welfare caused by the change in allocation, takes the form

$$K \sum_{i=1}^m p \cdot \, \mathrm{d}x_{ih}$$

for some $K > 0$. The welfare change caused by changes in health index for each individual can be can then assess the welfare change caused by changes in health as

$$\sum_{i=1}^m S_i' \frac{\partial u_i}{\partial h_i} \, \mathrm{d}h_i = H \sum_{i=1}^m \, \mathrm{d}h_i$$

for some $H > 0$, where $u_i(x_i, h_i)$ expresses the utility of individual i at bundle x_i and health state h_i. Taken together, we get the expression

$$\mathrm{d}S = K \sum_{i=1}^m p \cdot \, \mathrm{d}x_{ih} + H \sum_{i=1}^m \, \mathrm{d}h_i,$$

Since there are two unknown constants in this expression, the welfare change cannot be assessed directly. However, different medical interventions can be compared according to the ratio between $\sum_{i=1}^m p\cdot \mathrm{d}x_{ih}$ and $\sum_{i=1}^m \mathrm{d}h_i$, provided that interventions can be scaled up or down arbitrarily. Considering $-\sum_{i=1}^m \mathrm{d}h_i$, as the *cost* of the intervention and $\sum_{i=1}^m \mathrm{d}h_i$, as the *effect*, one gets in this case that interventions with smaller cost-effectiveness ratio are better from a welfare point of view.

(5) The economy $\mathcal{E} = ((X_i, P_i)_{i=1}^m, \omega)$ may be changed to an economy with private ownership $\mathcal{E}^p = (X_i, P_i, \omega_i)_{i=1}^m$ by assigning each individual the initial endowment $\omega_i = \dfrac{1}{m}\omega$. By Theorem 1.1, a Walras equilibrium $(x_1, \ldots, x_m, p)$ exists in $\mathcal{E}^p$.

The allocation $(x_1, \ldots, x_m)$ is envy-free: For each i, x_i is maximal for P_i when subjected to the budget constraint $p\cdot x_i \leq \dfrac{1}{m}p\cdot\omega$ which is the same for all individuals. It is Pareto optimal by Theorem 3.2. We have thus found that $\mathcal{E}$ has fair allocations.

(6) The dual problem has the form

$$\begin{aligned}
&\min v\\
&\sum_{r=1}^s u_r y_{0r} \geq 1\\
&v - \sum_{r=1}^s u_r y_{jr} \geq 0,\ j = 1, \ldots, n,\\
&v, u_1, \ldots, u_s \geq 0
\end{aligned}$$

which may be simplified to the equivalent minimization problem

$$\begin{aligned}
&\min v\\
&\sum_{r=1}^s u_r y_{0r} = 1\\
&\sum_{r=1}^s u_r y_{jr} \leq v,\ j = 1, \ldots, n,\\
&v, u_1, \ldots, u_s \geq 0
\end{aligned}$$

from which it is seen that u_r can be seen as prices of output, normalized so that total output value of the considered unit is 1. Minimization of v means that the prices are chosen so as to make the performance of the reference units considered as poor as possilble, or equivalently, to make the performance of the unit considered as good as possible.

(7) The Russell productivity index takes the value 1 at an efficient production: Indeed, let $x \in L$ be efficient, then an input vector x'' with $x'_h \leq x_h$ for all h and $x'_k < x_k$ for some k cannot belong to L, so that the only feasible vector $(\lambda_1, \ldots, \lambda_l)$ with $(\lambda_h x_h)_{h=1}^l \in L$ is $(1, \ldots, 1)$, and we get that $\lambda_R(x, L) = 1$.

Conversely, if $\lambda_R(x, L) = 1$, then the sum of λ_h to be used for computing $\lambda_R(x, L)$ must be l, so that $(\lambda_1, \ldots, \lambda_l) = (1, \ldots, 1)$. It then follows that x must be efficient.

Since there are sets L with inefficient input combinations x for which the Farrell index is 1, we have that $\lambda_F(x, L) \neq \lambda_R(x, L)$ for all such x and L.