Exercises to Chapter 5

(1) Let (x^0, p^0) be a Walras equilibrium in $\mathcal{E}$. For each i, the associated net trade $z_i^0 = x_i^0 - \omega_i$ belongs to the set $H_{p^0} = \{z' \mid p \cdot z' \leq 0\}$, and $z_i^0 + \omega_i$ is maximal for P_i on the set of all $z_i \in H_{p^0}$ such that $z_i + \omega_i \in X_i$. For $i, j \in \{1, \ldots, m\}$, $i \neq j$, we therefore cannot have that $z_i + \omega_\in P_i(z_j + \omega_i)$, and it follows that the net trades $(z_1^0, \ldots, z_m^0)$ are fair.

For the converse, a trivial example is the zero net trade with $z_i = 0$ for all i. We exclude this case in the sequel. Now consider an economy $(\mathbb{R}^2, P, (10, 10))_{i=1}^3)$ and the net trades $((-8, 8), (2, -3), (6, -5))$, where $P(x) = \{x\} + \mathbb{R}^2_{++}$ for $i = 1, 2, 3$. The net trades are fair but no halfspace contains all three net trades, so the associated allocation cannot be a Walras equilibrium.

For the infinite economy, we need to make precise what constitutes a system of net trades, more specifically the notion of aggregate feasibility, corresponding to the condition $\sum_{i=1}^M z_i = 0$ in the finite economy, and we take the notion of feasibility to mean that $0 \in A(z)$ (as in (1), p.143). Also, fairness of net trades with infinitely many agents should understood in the sense that no finite coalition would be better off having the net trade of any disjoint finite subset of the other agents. Finally, we assume that there is no countable subset I' if I such that $z_i = 0$ for all $i \in I \backslash I'$.

Consider the set $B(z)$ defined in (2), p.144. If $0 \in A(z)$, then also $0 \in B(z)$: Indeed, any $H \in \mathcal{F}$ has a superset H' with $\left\|\sum_{i \in H'} z_i\right\| < \varepsilon/2$, and $H \cup H'$ has a superset H'' with $\left\|\sum_{i \in H''} z_i\right\| < \varepsilon/2$, so that $H''' = H'' \setminus (H \cup H')$ is disjoint from H and satisfies $\left\|\sum_{i \in H'''} z_i\right\| < \varepsilon$.

We show that $B(z)$ does not intersect $\mathbb{R}^l_{--}$. Assume to the contrary that $u \in B(z) \cap \mathbb{R}^l_{--}$, and choose $b \in B$ arbitrarily. Then there is $H \in \mathcal{F}$ with $\left\|b - \sum_{i \in H'} z_i\right\| < \varepsilon/2$. Since also $b - u$ belongs to B, there is $H' \in \mathcal{F}$ with $H \cap h' \neq \emptyset$ and $\left\|(b - u) - \sum_{i \in H'} z_i\right\| < \varepsilon/2$, and it follows that H would be better off with the net trade of H', contradicting fairness.

By separation of $\operatorname{conv} B(z)$ and $\mathbb{R}^l_{--}$, we have that there is $p \in \mathbb{R}^l$ and an at most countable subset I' og I such that for all $i \in I \backslash I'$, $p \cdot z_i = 0$, and $z_i + \omega_i$ is maximal for P_i on all $z' + \omega_i$ with $p \cdot z' \leq 0$ and $z' \in B(z) - \mathbb{R}^l_+$. We thus get that $(z_i + \omega_i)_{i \in I}$ is a *restricted* Walras equilibrium.

To conclude from this that $(z_i + \omega_i)_{i \in I}$ is a Walras equilibrium, we need to assume

(i) *divisibility* of $B(z)$: $\operatorname{conv} B(z) = B(z)$

(ii) *full dimensionality* of $B(z)$: $\operatorname{conv} B(z) = \{z' \mid p \cdot z = 0$

(2) A family of exchanges $\mathfrak{x}^1, \dots, \mathfrak{x}^r$ gives rise to a bundles $x_1, \dots, x_m$ defined by

$$x_i = \omega_i + \sum_{k=1}^{r} \sum_{j=1}^{m} \mathfrak{x}^k(i,j), \tag{1}$$

and summation over all i gives

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{k=1}^{r} \sum_{i=1}^{m} \sum_{j=1}^{m} \mathfrak{x}^k(i,j) = \sum_{i=1}^{m} \omega_i,$$

where we have used that $\mathfrak{x}^k(i,j) + \mathfrak{x}^k(j,i) = 0$ and $\mathfrak{x}^k(i,i) = 0$ for all $i, j \in \{1, \dots, m\}$ and $k \in \{1, \dots, r\}$.

The family $\{\mathfrak{x}^1, \dots, \mathfrak{x}^r\}$ is an exchange equilibrium if

(1) the allocation $(x_1, \dots, x_m)$ determined by (1) satisfies $x_i \in X_i$, $i = 1, \dots, m$,

(2) there is no coalition H from $\{1, \dots, m\}$ with $i \in H$ and exchange $\mathfrak{x}$ with $\mathfrak{x}(h,j) = 0$ if $h \notin H$ or $j \notin H$ such that for each $i \in H$

$$x_i + \sum_{j \in H} \mathfrak{x}(i,j) - \sum_{(k,j) \in K_i} \mathfrak{x}^k(i,j) \in P_i(x_i) \tag{2}$$

for some subset K_i of $\{1, \dots, r\} \times [\{1, \dots, m\} \setminus H]$.

Let (x, p) be a Walras equilibrium in $\mathcal{E}$ and assume that the family $\{\mathfrak{x}^1, \dots, \mathfrak{x}^r\}$ satisfies $p \cdot \mathfrak{x}^k(i,j) = 0$ for all i, j and k and gives rise to the allocation x [this latter condition is missing in the statement of the problem]. Then for each i, any bundle x_i' obtainable as in (3) satisfies $p \cdot x_i' = p \cdot x_i$, consequently $x_i' \notin P_i(x_i)$, so that $\{\mathfrak{x}^1, \dots, \mathfrak{x}^r\}$ is indeed an exchange equilibrium.

To show that an exchange equilibrium $\{\mathfrak{x}^1, \dots, \mathfrak{x}^r\}$ with the property $p \cdot \mathfrak{x}^r(i,j) = 0$ for all i, j and r does not necessarily give rise to a Walras equilibrium we may choose an economy $\mathcal{E}$ with $m = 2$ and an allocation in the core of $\mathcal{E}$ but not a Walras equilibrium allocation, and define the exchange $\mathfrak{x}$ by $\mathfrak{x}(i,j) = x_i - \omega_i$ for $i \neq j$. It is easily checked that $\mathfrak{x}$ is an exchange equilibrium. Choosing p such that $p \cdot \mathfrak{x}(1,2) = 0$, we have that the desired counterexample.

For the infinite economy, an exchange equilibrium is a finite family of exchanges $\mathfrak{x}^k : I \times I \to \mathbb{R}^l$ with $\mathfrak{x}^k(i,i') = -mathfrakx^k(i',i)$ for all $i, i' \in I$, $k = 1, \dots, r$, such that

(1) the allocation $(x_i)_{i \in I}$ is feasible in the sense of (1)p.143, and $x_i \in X_i$, $i \in I$,

(2) for each $J \in \mathcal{F}$, there is no coalition $H \in \mathcal{F}$ and exchange $\mathfrak{x}$ with $\mathfrak{x}(h,j) = 0$ if $h \notin H$ or $i' \notin H$ such that for each $i \in H$

$$x_i + \sum_{j \in H} \mathfrak{x}(i,j) - \sum_{(k,j) \in K_i} \mathfrak{x}^k(i,j) \in P_i(x_i) \tag{3}$$

for some subset K_i of $\{1,\dots,r\}\times[\{1,\dots,m\}\setminus H\cap J]$.

To see that the allocation associated with an exchange equilibrium is a Walras equilibrium, we need only to notice that the allocation belongs to the core of $\mathcal{F}$. Indeed, if $(x_i)_{i\in I}$ had an improvement via some coalition H, then this improvement could be implemented by an exchange involving only members of H, and without canceling any exchange transactions with individuals not in H. We now get the desired conclusion from Thm.5.1.

(3) We assume that the measure space $(A,\mathcal{A},\lambda)$ is nonatomic (otherwise the statement would not necessary be true). Let ξ be the net trade corresponding to x and suppose that ξ can be improved by the coalition $S\in\mathcal{A}$, so that $\eta(S)\in P_S(\xi)$ for some η with $\eta(S)=0$. Consider now the net trade $\hat{\xi}$ defined by

$$\hat{\xi}(T)=\begin{cases}(\xi(T),\lambda(T)) & T\in\mathcal{A},T\subseteq S,\\ 0 & \text{otherwise}\end{cases}$$

We extend preferences to $(l+1)$-dimensional net trades in the trivial way meaning that preferences are independent of the $(l+1)$st component. Now we use that $\mathcal{P}(\hat{\xi})$ is convex (see proof of Thm.5.3), and since it contains $(0,\lambda S)$ and $(0,0)$, it contains also $(0,\varepsilon)$ for $0\le\varepsilon<\lambda(S)$, meaning that there is some net trade η' and coalition T with $\eta'\in P_T(\xi)$, $\eta'(T)=0$, and $\lambda(T)=\varepsilon$, which gives us the desired improvement.

(4) [There is a typo in the definition of $\sigma_i(p)$, which should be

$$\{x_i'\in\operatorname{cl}P_i(x_i)\mid p\cdot x_i''\ge p\cdot x_i',\ \text{all}\ p\cdot x_i'\in\operatorname{cl}P_i(x_i)\},$$

the set of cost minimizers among bundles no worse than x_i]

Suppose that $-(l-1)(\alpha,\alpha,\dots,\alpha)$ belongs to the interior of

$$\operatorname{conv}\left(\sum_{i=1}^{m}[\sigma_i(x_i)-\{\omega_i\}\cup\{0\}]\right),$$

then there is also a hyperplan with $-(l-1)(\alpha,\alpha,\dots,\alpha)$ in the interior of its intersection with $\operatorname{conv}(\sum_{i=1}^{m}[\sigma_i(x_i)-\{\omega_i\}\cup\{0\}])$. Using the Shapley-Folkman theorem, we can write $-(l-1)(\alpha,\alpha,\dots,\alpha)$ as a sum of points $z_i\in\operatorname{conv}([P_i(x_i)-\{\omega_i\}]\cup\{0\}])$ and at most $l-1$ not in $P_i(x_i)-\{\omega_i\}$. But then there would be an improvement of x via some subset of S, a contradiction, since x belongs to the core of $\mathcal{E}$.

It follows that there is $p\in\triangle$ such that $p\cdot(x_i'-\omega_i)\ge-(l-1)p\cdot(\alpha,\dots,\alpha)=-(l-1)\alpha$ for all i, which gives the desired conclusion.

(5) $C(K)$ is admissible, since it satisfies conditions (a) and (b) of Thm.5.11. By the same reasoning, one has that BV is admissible. $L^p(\mu)$ is not admissible: It is a vector lattice, but not a Kakutani space, as condition (i) on p.174 fails, and consequently, by Lemma 5.5, it cannot be admissible. The same argumentation holds for ba($\mathbb{A}$).

(6) An allocation in this economy is a bijection $\sigma : K \to K$, and a price is a map $p : K \to \mathbb{R}_+$, the pair (σ, p) is a Walras equilibrium if $p(\sigma(i)) \leq p(i)$ and $P_i(\sigma(i)) \cap p^{-1}(\{j \mid p(j) \leq p(i)\}) = \emptyset$, for all $i \in K$, where for each $i, j \in K$, $P_i(j)$ is the set of commodities preferred to j by i.

Existence of a Walras equilibrium can be obtained only under simplifying assumptions, for example if there is only a finite set of types, whereby a type includes the commodity as well as the preferences of the owner of this commodity. Standard exstence theorems do not apply due to indivisibility.