**Exercises to Chapter 6**

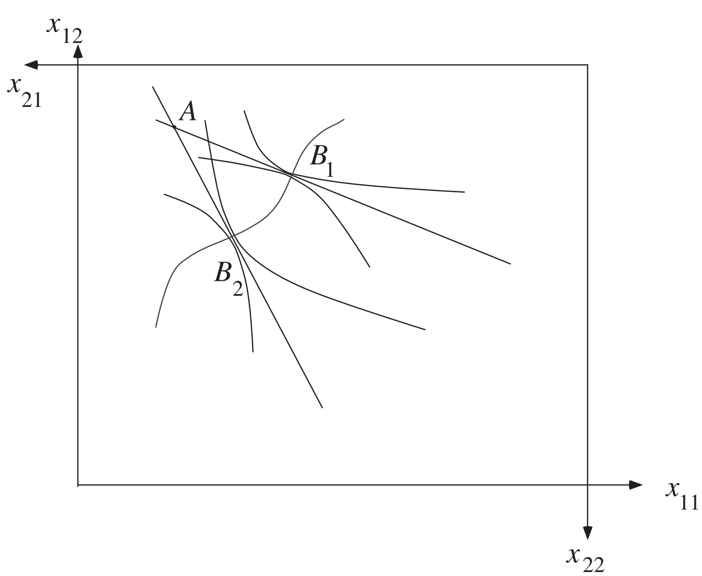
**(1)** For every economy $\mathcal{E} = (X_i, P_i, \omega_i)_{i=1}^m$, there is an economy with the measure space $[0, 1[$ endowed with Lebesgue measure, such that $(X_t, P_t, \omega_t) = (X_i, P_i, \omega_i)$ for $t \in \left[\frac{i-1}{m}, \frac{i}{m}\right]$. Clearly, a Walras equilibrium $(x, p)$ in $\mathcal{E}$ induces a Walras equilibrium in the infinite economy, since

$$\int_0^1 x(t)\,\mathrm{d}t = \sum_{i=1}^m \frac{1}{m} x_i = \frac{1}{m}\sum_{i=1}^m x_i = \frac{1}{m}\sum_{i=1}^m \omega_i = \int_0^1 \omega(t)$$

and for each $t$, $x(t)$ is individually optimal given $p$.

In view of this, it is clear that there are atomless economies with more than one Walras equilibrium (for example, the atomless version of the economy depicted in Box 1).

**(2)** The figur in Box 1, reproduced below, can be used to express the case considered: Assume that initial endowment in the box is at a point $C$ on the continuation of th e line segment from $B_2$ to $A$ beyond $A$. Then $B_2$ is a Walras equilibrium, but if $C$ is moved to a point $D$ southwest of $C$ on the line through $A$ and $B_1$, the point $B_1$ will be Walras equilibrium strictly preferred to $B_2$ by consumer 1.




The situation generalizes to more than two commodities and agents: Suppose that in the economy $\mathcal{E} = (\mathbb{R}^l_+, P_i, \omega_i)_{i=1}^m$, there are two consumers, say 1 and 2, such that if 1 transfers to 2 some amount of all commodities before trading in the market, thus changing the endowment from $\omega_1$ to $\omega_1'$, then consumer 1 will be better off in the resulting Walras equilibrium. Let $p$ and $p'$ be Walras equilibrium prices without and with the transfer, respectively.

Let $\mathcal{A}$ be a 2-dimensional affine subspace containing $\omega_1$, $\omega_1'$, and $\omega_1 + \omega_2$. Then intersections with $\mathcal{A}$ of the hyperplanes $\{x \mid p \cdot x = p \cdot \omega_1\}$ and $\{x \mid p' \cdot x = p \cdot \omega_1'\}$ are straight lines intersecting each other in some point $A$. Let $A$ define the endowment $\omega_1''$ of consumer 1, and let $\omega_2'' = (\omega_1 + \omega_2) - \omega_2''$. Then both $p$ and $p'$ with be Walras equilibrium prices af the economy $\mathcal{E}''$ where $\omega_i$ has been replaced by $\omega_i''$ for $i = 1, 2$.

[There are several contributions to the literature, following Chichilnisky (1980), which show that a local version of the transfer paradox can be occur even when the Walras equilibrium is locally unique.]

**(3)** By symmetry it is enough to show that aggregate demand for one commodity, say commodity 1, increases when the price of another commodity, say commodity 2, is increased.

We find the demand of consumer 1, by maximizing utility $2^{3/2}\sqrt{x_1} + \sqrt{x_2} + 2^{3/2}\sqrt{x_3} + \sqrt{x_4}$ under the budget constraint $\sum_{h=1}^4 p_h x_h = \sum_{h=1}^4 \omega_{1h}$. The first order conditions are

$$\frac{\sqrt{2}}{\sqrt{x_1}} = \lambda p_1,\ \frac{1}{2}\frac{1}{\sqrt{x_2}} = \lambda p_2,\ \frac{\sqrt{2}}{\sqrt{x_3}} = \lambda p_3,\ \frac{1}{2}\frac{1}{\sqrt{x_4}} = \lambda p_4,$$

which may be rewritten as

$$x_1 = \frac{2}{\lambda^2 p_1^2},\ x_2 = \frac{1}{4\lambda^2 p_2^2},\ x_3 = \frac{2}{\lambda^2 p_3^2},\ x_2 = \frac{1}{4\lambda^2 p_2^2}.$$

Using the budget constraint we find that

$$\lambda^2 = \frac{\dfrac{2}{p_1} + \dfrac{1}{4p_2} + \dfrac{2}{p_3} + \dfrac{1}{4p_4}}{p_1\omega_{11} + p_2\omega_{12} + p_3\omega_{13} + p_4\omega_{14}}.$$

We can now find the derivative of the demand for commodity 1 w.r.t. $p_2$ as

$$\frac{\partial \xi_{11}}{\partial p_2} = \frac{\mathrm{d}x_1}{\mathrm{d}\lambda^2}\frac{\mathrm{d}\lambda^2}{\mathrm{d}p_2} = \left(-\frac{2}{p_1^2\lambda^4}\right)\frac{\mathrm{d}\lambda^2}{\mathrm{d}p_2},$$

and since the derivative of $\lambda^2$ w.r.t. $p_2$ is negative (easily checked by performing the differentiation in (**??**)), we get that $\dfrac{\partial \xi_{11}}{\partial p_2} > 0$. Repeating the argument for consumer 2, one gets the desired result.

**(4)** We show that $\zeta$ satisfies the following condition used in the proof of Theorem 6.5:

If $p^0$ is an equiliibrium price, and $p(t)\cdot\zeta(p^0)=0$ and $p^1\neq p^0$, then $p^0\cdot\zeta(p^1)>0$.

We consider first the case $l=2$. Changing units if necessary, we may assume that $p^0=(1,1)$ so that $\zeta_1(p^0)=\zeta_2(p^0)$, and multiplying $p^1$ by a scalar and changing labels of commodities if necessary we may assume that $p_1^1=1$, $p_2^1>1$. Increasing $p_2$ from 1 we have by gross substition that $\dfrac{\mathrm{d}\zeta_1}{\mathrm{d}p_2}>0$, and since Walras' law must be satisfied at all $p$, we have that $\dfrac{\mathrm{d}\zeta_2}{\mathrm{d}p_2}>0$, but since $p_2$ has become larger than $p_1$, the numerical value of $\dfrac{\mathrm{d}\zeta_1}{\mathrm{d}p_2}>0$ must exceed that of $\dfrac{\mathrm{d}\zeta_2}{\mathrm{d}p_2}>0$, so that

$$\zeta_1(p^0)+\frac{\mathrm{d}\zeta_1}{\mathrm{d}p_2}\,\mathrm{d}p_2>\zeta_2(p^0)+\frac{\mathrm{d}\zeta_2}{\mathrm{d}p_2}\,\mathrm{d}p_2.$$

Repeating the argument for arbitrary $p_2\in[1,p_2^1]$, we may conclude that $p^0\cdot\zeta(p^1)>0$.

For $l>2$ a similar argumentation can be carried out, it is however rather lengthy, instead we refer to Arrow, Block and Hurwicz (1959).

**(5)** For the problem to be meaningful, we assume that all $p^k$ are different. Choose a system of open sets $(U_{p^k})_{k=1}^r$ in $\triangle$ such that $p^h\notin U_{p^k}$ for $h\neq k$, each $k$, such that $\triangle$ is covered by the family $(U_{p^k})_{k=1}^r$, and let $(\psi_k)_{k=1}^r$ be a continuous partition of unity subordinated this covering (i.e., each $\psi_k$ is a continuous map from $U_{p^k}$ to $[0,1]$, and $\sum_{k=1}^r\psi_k(p)=1$ for each $p\in\triangle$. Then the map $f:\triangle\to\mathbb{R}^l$ defined by

$$f(p)=\sum_{k=1}^r\psi_k(p)z^k$$

is continuous and satisfies $p\cdot f(p)=0$ for all $p\in\triangle$, and the graph of $f$ contains the points $(p^k,z^k)$ for $k=1,\dots,r$. Now an application of Theorem 6.6 gives the desired result.

**(6)** Actually Newton's method works well in the case considered: The Jacobian of the function is

$$\begin{pmatrix}2x_1 & 2x_2 & 2x_3\\ 2x_1 & 2x_2 & -1\\ 1 & 1 & 1\end{pmatrix},$$

which assessed at $x=(1,0,1)$ gives the matrix

$$J=\begin{pmatrix}2 & 0 & 2\\ 2 & 0 & -1\\ 1 & 1 & 1\end{pmatrix}\text{ with inverse }J^{-1}=\begin{pmatrix}\frac{1}{6} & \frac{1}{3} & 0\\ -\frac{1}{2} & 0 & 1\\ \frac{1}{3} & -\frac{1}{3} & 0\end{pmatrix}.$$

The value of the function at $(1,0,1)$ is $(-1,-1,-1)$ and the first step can be found by multiplying this vector by the matrix $J^{-1}$, giving the first step $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, adding this steo to $(1, 0, 1)$ gives the new point $\left(\frac{3}{2}, \frac{1}{2}, 1\right)$. The procedure can now be repeated at this point to define a sequence of points converging to $(1, 1, 1)$ which is indeed a root of the system of equations.

The method may will fail at other initial values, e.g. for $x = (0, 0, 0)$, where the Jacobian is singular.

**References**

Arrow, K.J., H.D.Block and L.Hurwicz (1959), On the Stability of the Competitive Equilibrium, II, Econometrica. 27, 82-109.

Chichilnisky, G. (1980), Basic goods, the effects of commodity transfers and the international economic order, Journal of Development Economics 7, 505-519.