**Exercises to Chapter 7**

**(1)** The demand function of consumer $i$ has the form

$$\xi_i(p, p\cdot\omega_i) = \left(\alpha_{i1}\frac{p\cdot\omega_i}{p_1}, \dots, \alpha_{il}\frac{p\cdot\omega_i}{p_l}\right),$$

for $i = 1, \dots, m$, so that the equilibrium manifold consists of all $(p, \omega_1, \dots, \omega_l)$ such that

$$\sum_{i=1}^{m} \alpha_{ih}\frac{p\cdot\omega_i}{p_h} = \sum_{i=1}^{m} \omega_{ih},\ h = 1, \dots, l. \tag{1}$$

If, as indicated by the notation, the coefficients $\alpha_{ih} = \alpha_h$ are independent of $i$, then this equation can be reduced to

$$\frac{\sum_{i=1}^{m} p_h\omega_{ih}}{p\cdot\sum_{i=1}^{m}\omega_i} = \alpha_h,\ h = 1, \dots, l.$$

**(2)** The demand function of consumer $i$ has the form

$$\xi_i(p, p\cdot\omega) = (p\cdot\omega_i)(a_{i1}, \dots, a_{il}),$$

so that the equilibrium manifold is the set

$$\left\{(p, \omega_1, \dots, \omega_m) \,\middle|\, \sum_{i=1}^{m}(p\cdot\omega_i)(a_{i1}, \dots, a_{il}) = \sum_{i=1}^{m}\omega_i\right\}.$$

With identical constants $(a_{i1}, \dots, a_{il}) = (a_1, \dots, a_l)$, $i = 1, \dots, m$, this becomes

$$\left\{(p, \omega_1, \dots, \omega_m) \,\middle|\, (a_1, \dots, a_l) = \frac{1}{p\cdot\sum_{i=1}^{m}\omega_i}\sum_{i=1}^{m}\omega_i\right\}.$$

**(3)** We find the no-trade equilibria corresponding to the point $(p^0, \omega_1^0, \dots, \omega_m^0)$: Since in a no-trade equilibrium, the demand equals the endowment, we must have that this endowment $\overline{\omega}_i^0$ must satisfy

$$\alpha_{ih}\frac{p^0 \cdot \omega_i^0}{p_h^0} = \overline{\omega}_{ih}^0, \text{ for } i = 1, \ldots, m, \ h = 1, \ldots, l,$$

since $p^0 \cdot \overline{\omega}_i^0 = p \cdot \omega_i^0$ for each $i$. A continuous path from $(p^0, \omega_1^0, \ldots, \omega_m^0)$ to $(p^0, \overline{\omega}_1^0, \ldots, \overline{\omega}_m^0)$ is given by

$$t \mapsto (p^0, t\overline{\omega}_1^0 + (1-t)\omega_1^0, \ldots, t\overline{\omega}_m^0 + (1-t)\omega_m^0)$$

for $t \in [0, 1]$. Clearly, each point on this curve belongs to the equilibrium manifold.

Similarly, there is a no-trade equilibrium corresponding to $(p^1, \omega_1^1, \ldots, \omega_m^1)$, namely that with endowments $\overline{\omega}_i^1$ such that

$$\alpha_{ih}\frac{p^1 \cdot \omega_i^1}{p_h^1} = \overline{\omega}_{ih}^1, \text{ for } i = 1, \ldots, m, \ h = 1, \ldots, l,$$

and $p^0 \cdot \overline{\omega}_i^0 = p \cdot \omega_i^0$ for each $i$, and a continuous path from $(p^1, \overline{\omega}_1^1, \ldots, \overline{\omega}_m^1)$ to $(p^1, \omega_1^0, \ldots, \omega_m^1)$ in the equilibrium manifold is given by

$$t \mapsto (p^1, t\omega_1^1 + (1-t)\overline{\omega}_1^0, \ldots, t\omega_m^1 + (1-t)\overline{\omega}_m^1).$$

It remains to find a continuous curve in the equilibrium manifold from $(p^0, \overline{\omega}_1^0, \ldots, \overline{\omega}_m^0)$ to $(p^1, \overline{\omega}_1^1, \ldots, \overline{\omega}_m^1)$. For $t \in [0, 1]$ let

$$p^t = tp^1 + (1-t)p^0,$$

and for each $i$, let $w_i^t = t(p^1 \cdot \omega_i) + (1-t)(p^0 \cdot \omega_i^0)$. Define

$$\overline{\omega}_i^t = \left(\alpha_{i1}\frac{w_i^t}{p_1^t}, \ldots, \alpha_{il}\frac{w_i^t}{p_l^t}\right)$$

for $i = 1, \ldots, m$. Then each $(p^t, \overline{\omega}_1^t, \ldots, \overline{\omega}_m^t)$ is a no-trade equilibrium, and

$$t \mapsto (p^t, \overline{\omega}_1^t, \ldots, \overline{\omega}_m^t)$$

defines a continuous curve in the equilibrium manifold from $(p^0, \overline{\omega}_1^0, \ldots, \overline{\omega}_m^0)$ to $(p^1, \overline{\omega}_1^1, \ldots, \overline{\omega}_m^1)$. Connecting the three curves found above we get the solution.

**(4)** There function given in the exercise is incompletely formulated, since excess demand of commodity 1 (2) is positive (negative) for all relevant $p$. A possible reformulation would be:

$$\zeta_1^\alpha(p) = \frac{p_2^\alpha - \frac{1}{2}}{p_1},\ \zeta_2^\alpha(p) = \frac{\frac{1}{2} - p_(2^\alpha}{p_2}$$

which satisfies Walras' law and has the property that $\zeta_h^\alpha$ becomes numerically large if $p_1$ or $p_2$ get close to 0. We have that if $p_2 \geq \left(\frac{1}{2}\right)^{\frac{1}{10}}$, then all excess demands of commodity 2 are nonpositive, and if $p_2 \leq \frac{1}{2}$, then all excess demands of commodity 2 are nonnegative.

We may now apply the Corollary to Theorem 7.3, using $\zeta^\alpha(p_2)$ as the excess demand function $\zeta(t)$, and choosing $n$ arbitrary values of $\alpha$ for each of the types in $C^0$. Then

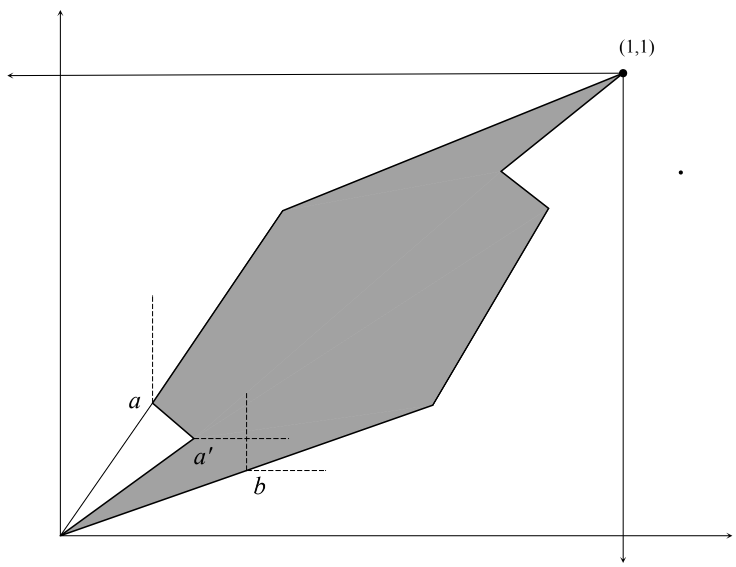
$$t_{\min} = \frac{1}{2},\ t_{\max} = \left(\frac{1}{2}\right)^{\frac{1}{10}},\ \text{and}\ \zeta'(t) = -\frac{t^\alpha(\alpha-1) - \frac{1}{2}}{t^2}$$

Assuming that the types differ by at least 1, we get that the values $\zeta_1(t), \ldots \zeta_n(t)$ differ by at least $t^\alpha - t^{\alpha-1}$, so that $\sqrt{\zeta_1(t)^2 + \cdots + \zeta_n(t)^2} > 2^{-10}$, and for $t > 1/2$ we have that $\zeta_i'(t) < 4 \cdot 9 = 36$. Inserting, we have a (crude) upper bound for $M$ amounting to $36 \cdot 2^{10}$, giving an upper bound for the expected number of equilibria of

$$\mathsf{E}\nu = \sqrt{n}\left[\left(\frac{1}{2}\right)^{\frac{1}{10}} - \frac{1}{2}\right] 36 \cdot 2^{10}.$$

Clearly, the bound could be improved with a better evaluation of $\sqrt{\zeta_1(t)^2 + \cdots + \zeta_n(t)^2}$.

**(5)** The unit isoquants of the two technologies are illustrated in the figure. If the price of commodity 1 relative to that of commodity 2 is high, the first industry will use technology $a'$, whereas $a$ is used when the relative price of commodity 1 is low. The result is a factor price equalization domain of somewhat irregular shape as indicated.




The probability of factor price equalization is then the area of the factor price equalization relative to that of the square from $(0,0)$ to $(1,1)$. Its numerical value will depend on the value assigned to the points $a$, $a'$ and $b$.