Exercises to Chapter 8

(1) We normalize prices setting $p_1 = 1$. The producer will then choose to supply the amount $\frac{1}{p_2}$ with input $\frac{1}{p_2^2}$, giving a profit $p_2\frac{1}{p_2} - \frac{1}{p_2^2}$. Then the consumers have demand for commodity 1 of size

$$\begin{aligned}&\frac{2}{5}\left[2 + 4p_2 + \frac{1}{3}\left(1 - \frac{1}{p_2^2}\right)\right],\\ &\frac{3}{4}\left[3 + 7p_2 + \frac{1}{3}\left(1 - \frac{1}{p_2^2}\right)\right],\\ &\frac{3}{7}\left[4 + 5p_2 + \frac{1}{3}\left(1 - \frac{1}{p_2^2}\right)\right],\end{aligned} \tag{1}$$

and in an equilibrium with positive production the sum of these demands should equal the net supply which is

$$9 - \frac{1}{p_2^2}.$$

There is however no nonnegative value of p_2 for which demand does not exceed supply, so we have to look for cases where $p_2 < 1$ so that profit is negative with positive production, meaning that in such cases $y_2 = y_1 = 0$. In this situation, the income term in the consumer demand (1) is zero, and the consumer demand reduces to

$$\begin{aligned}&\frac{2}{5}\left[2 + 4p_2\right],\\ &\frac{3}{4}\left[3 + 7p_2\right],\\ &\frac{3}{7}\left[4 + 5p_2\right],\end{aligned}$$

the sum of which equals the supply 9 at the price $p_2 = 0.471$.

(2) Let the Cobb-Douglas utility function of consumer i be $u_i(x_i) = \prod_{h=1}^{l}(x_{ih})^{\alpha_{ih}}$, giving rise to the demand function

$$\xi_i(p, \omega_i) = \left(\alpha_{ih}\frac{p \cdot \omega_i}{p_h}\right)_{h=1}^{l}.$$

Normalized Walras equilibrium prices $p \in \triangle$ for $\mathcal{E}$ satisfies the equation system

$$\begin{aligned} &\sum_{i=1}^{m} \alpha_{ih} \frac{p \cdot \omega_i}{p_h} = \sum_{i=1}^{m} \omega_{ih}, h = 1, \dots, l, \\ &\sum_{h=1}^{l} p_h = 1. \end{aligned} \tag{2}$$

To show that there is a unique equilibrium we check that individual demand functions satisfies gross substitution: Indeed, let i be arbitrary, and let $h, k \in \{1, \dots, l\}$, $h \neq k$, then

$$\frac{\partial}{\partial p_k} \left(\alpha_{ih} \frac{p \cdot \omega_i}{p_h} \right) = \alpha_{ih} \frac{\omega_{ik}}{p_h} > 0.$$

Since the equations system (2) has a unique solution in $\triangle$, we may consider this solution as a function F of $\omega = (\omega_1, \dots, \omega_m) \in \mathbb{R}^{ml}$. The game of withholding some resources from the market has then strategy spaces $\Sigma_i = \{\omega'_i \in \mathbb{R}^l_+ \mid \omega'_{ih} \leq \omega_{ih}, h = 1, \dots, l\}$ for $i = 1, \dots, m$, and the payoff function of consumer i given the strategy array $\omega' = (\omega'_1, \dots, \omega'_m)$ is

$$\pi_i(\omega') = u_i(\xi_i(F(\omega'), \omega'_i) + (\omega_i - \omega'_i)),$$

for $i = 1, \dots, m$. A strategy array ω^0 is a Nash equilibrium of the game if

$$\pi_i(\omega^0) \geq \pi_i(\omega'_i, \omega^0_{)i(}), \text{ all } \omega'_i \in \Sigma_i, i = 1, \dots, m,$$

where $(\omega'_i, \omega^0_{)i(})$ is the strategy array obtained from ω^0 after replacing ω^0_i by ω'_i.

Existence of a Nash equilibrium cannot be inferred from standard results, and indeed standard examples show that Nash equilibria do not always exist: Consider an economy with $m = l = 2$ and utilitiy functions

$$u_1(x_1) = x_{11}^{1/3} x_{12}^{2/3},\ u_2(x_2) = x_{21}^{2/3} x_{22}^{1/3}, \omega_1 = \omega_2 = (1, 1).$$

If consumer 1 sends $\lambda_1 \in [0, 1]$ of commodity 1 and $\lambda_2 \in [0, 1]$ to the market, withholding the rest, then the equilibrium equation for commodity 1 becomes

$$\frac{1}{3} \frac{\lambda_1 p_1 + \lambda_2 (1 - p_1)}{p_1} + \frac{2}{3} \frac{1}{p_1} = 1 + \lambda_1$$

which gives the equilibrium prices

$$p_1 = \frac{1}{1 + \frac{1}{3}\lambda_2 + \frac{2}{3}\lambda_1}, p_2 = 1 - p_1.$$

After inserting numerical values of λ_1 and λ_2, it turns out that final utility of consumer

1 increases as more and more of commodity 2 is withheld from the market, meaning that there is no optimal strategy for consumer 1 for the given strategy of consumer 2. Since the endowment of consumer 2 enters only as a constant, we have that there is no Nash equiliibrium in this game.

If in the general case $(\omega_1^0, \dots, \omega_m^0)$ is itself a Nash equilibrium, the resulting allocation is trivially Pareto optimal. Otherwise, we would have an allocation obtained in the market different from the Walrasian where marginal rates of substitution are the same for all individuals, and a final allocation obtained by adding the the withheld amounts to the bundles of each consumer, and then marginal rates of substitution cannot be expected to be the same for all.

(3) As in Exercise (1), we normalize prices so that $p_1 = 1$. If the producer chooses the output y_2, then input is y_2^2 and profit is $p_2 y_2 - y_2^2$. The demand for commodity 1 of the three consumers is

$$\frac{2}{5}\left[2 + 4p_2 + \frac{1}{3}\left(p_2 y_2 - y_2^2\right)\right],$$
$$\frac{3}{4}\left[3 + 7p_2 + \frac{1}{3}\left(p_2 y_2 - y_2^2\right)\right],$$
$$\frac{3}{7}\left[4 + 5p_2 + \frac{1}{3}\left(p_2 y_2 - y_2^2\right)\right],$$

and the sum should equal supply which is $9 - y_2^2$, which for given y_2 is solved to give the equilibrium price p_2. The value of y_2 is then selected such that profits at the resulting equilibrium prices is maximal.

Numerical computations give the following table:

y_2	p_2	profit
1.5	0.325	-1.7625
1	0.395	-0.605
0.5	0.445	-0.0275
0.25	0.46	0.0525
0.15	0.47	0.048

We conclude that in the Cournot-Walras equilibrium output is 0.25, and the resulting price system is $(1, 0.46)$.

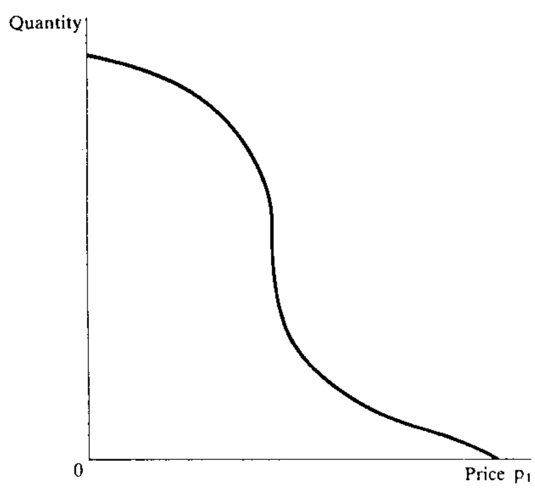
(4) When taking replica, the number of firms increase with n, but so does the number of consumers. and thereby the market. This means that the situation facing the individual firm in the nth replica looks much the same as that of the firm in $\mathcal{E}$. However, there is a difference: Each firm contemplating a deviation will face a residual market – or more correctly, will expect to influence the established market equilibrium prices – which is made up by many more different consumers with possibly different response, and intuitively this will make deviation from an established allocation more attractive.

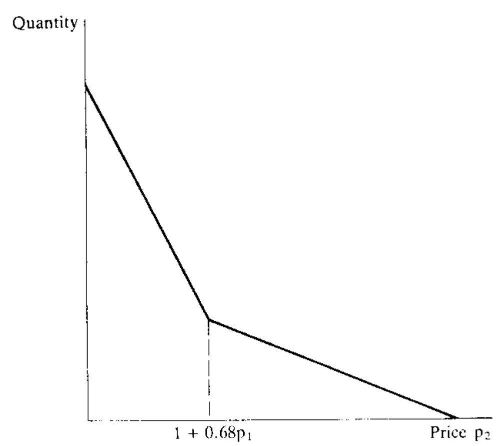
This intuitive version of this situation is sustained by the classical model of Cournot oligopoly in an economy with two goods, where the consumer side is described by a demand schedule, and in equilibrium, each firm is facing the residual demand given the choices of the other firms. Suppose that initially there are n firms and the demand function has the form $p = 1 - x$, where x is total output. In the symmetric equilibrium with zero cost each firm produces $\frac{1}{m+1}$ and total output is $\frac{m}{m+1}$. In the k-replica, the demand function is $p = 1 - \frac{1}{k}x$, the km firms each produce $\frac{k}{km+1}$, and total output satisfies

$$\frac{k^2 m}{km+1} = \frac{km}{m+\frac{1}{k}} \to k$$

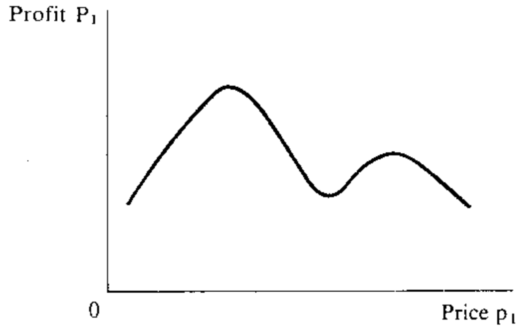
showing that the allocation tends towards the Walras equilibrium allocation as k grows large.

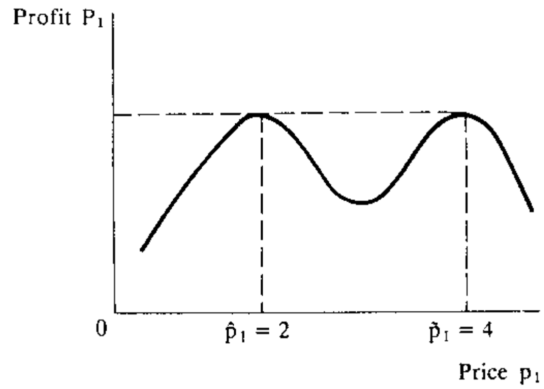
(5) The answers depend on a numerical analysis of the given functions. The graphical illustrations are taken from the article by Bonanno (1990). The first two figures illustrate the demand functions for firm 1 (left) and 2 (right) for given value of p_2 (p_1)

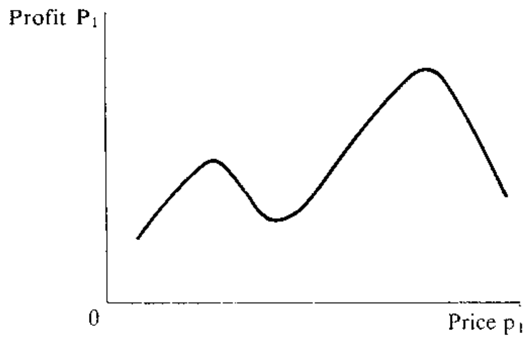




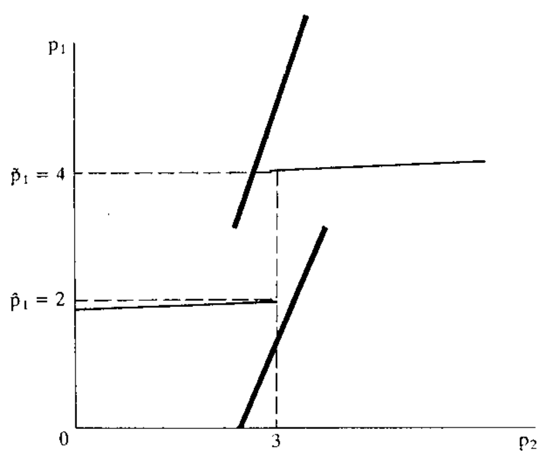
The next figure shows the profit function of firm 1 for $p_2 < 3$, $p_2 = 3$, $p_2 > 3$. It is seen that the profit function cannot be quasi-concave in p_1 and p_2.







Finally, the reaction curves are given below (thin for 1, fat for 2), and they do not intersect, so that there is no equilibrium.



(6) [Missing term in expression (6) which should be $\sum_{h=1}^{l-1} a_{lh} y_h = L(p)$]Assuming that each capitalist is in charge of the output of exactly one commodity, being the only producer of this commodity, we get that the profit of the $j$the capitalist at prices p and output y is found as the value of output $p_j y_j$ minus the cost of producing this output, which amounts to $\sum_{h=1}^{l-1} p_h(a_{hj} y_j)$, minus labor cost $1 \cdot a_{lj} y_j$, giving the expression in (4).

Let p be given, and let F and G be continuous. For each $y = (y_1, \dots, y_{l-1})$ with $\sum_{h=1}^{l-1} y_h = L(p)$, the left-hand side in (5), $F(p) + G(p, \pi(y))$ may be taken as belonging to $\mathbb{R}_+^{l-1}$, and assuming that A is productive, Lemma 2.2 of Ch.2 gives that

$$(I-A)^{-1}\left[F(p)+G(p,\pi(y)\right]$$

sends y to an element of $\mathbb{R}_+^{-1}$. This means that we get a map from $\{y \in \mathbb{R}_+^{l-1} \mid \sum_{h=1}^{l-1} y_h =$ $L(p)$ to itself possibly after normalizing, and by Brouwer's fixed point theorem, this map has a fixed point p^0. Assuming now that F and $G(p,\pi)$ satisfy the natural properties of demand functions, that is

$$p \cdot F(p) = L(p),\ p \cdot G(p,\pi) = \pi(y) = p(I-A)y - \sum_{h=1}^{l} a_{lh} y_h$$

for all p and y, we get that y^0 solves (5) and (6) [corrected version]. Clearly, the map taking p to this y^0 expresses the demand for production (for consumption as well as for inputs) which will balance the market.

Assuming that for each p, a solution $y(p)$ has been selected, one can state a Bertrand-Nash equilibrium in this economy as an array $p^0 = (p_1,\ldots,p_{l-1})$ of prices such that for each j,

$$\pi_j(y(p^0)) \geq \pi_j(y(p'_j, p^0_{)j(})$$

for all $p' \in \mathbb{R}_+$.