Exercises to Chapter 9

(1) A trivial game form $G = ((S_i)_{i=1}^m, \mathbb{R}^l, f)$, where individuals choose net trades z_i (so that $S_i = \mathbb{R}^l$) and the outcome function sends every array of net trades into the zero net trade array,

$$f(z_1, \dots, z_n) = (0, \dots, 0),$$

can be used, at least if the notion of implementation is understood such that the game form may prescribe *net trades* in the economy (as it does for strategic market games). Clearly, in the game $G[\mathcal{E}]$ nothing happens, every individual remains with the initial endowment, so that the result is indeed individually rationel.

(2) The game form described will not necessarily result in Walras equilibrium allocations, since there is no rule for deciding how to distribute profit from producers to consumers. So we consider only cases where such transfers of profits are not relevant.

In the special case of constant returns to scale, a Walras equilibrium (x, y, p) allocation can be obtained as Nash equilibrium: Every player, consumer or producer, announces the equilibrium price p, consumer i announces x_i, $i = 1, \dots, m$, and producer j announces y_j. Then $M(p)$ consists of all players, and it is easily checked that no player can obtain an improvement by selecting an alternative strategy (p', x'_i) or (p', y'_j).

For the converse, we may identify the producers with a consumer having consumption set $-Y_j$, endowment 0 and preferences P_j given by $P_j(y_j) = \{y'_j \in Y_j \mid y'_j \leq y_j\}$, for $j = 1, \dots, n$. Then Theorem 9.1 applies, so that if $m + n \geq l + 1$, then every Nash equiliibrium in the game gives rise to a Walras equilibrium.

(3) We consider here the case where the assumption $\bar{x}_i = f_i(s) \in \mathbb{R}^l_{++}$ in Theorem 9.2 is not necessarily fulfilled for all i, for s a Nash equilibrium of $G(\mathcal{E})$.

As in the proof of Theorem 9.2, we have that the allocation $(\bar{x}_1, \dots, \bar{x}_m)$ is Pareto optimal, and arguing as in the proof of Theorem 3.2, we get that there is a price vector $p \in \mathbb{R}^l_{++}$ such that $\bar{x}_i$ minimizes expenditure $p \cdot x_i$ for all $x_i \in \mathrm{cl}P_i(\bar{x}_i)$. Assuming that $\sum_{i=1}^m \omega_i \in \mathbb{R}^l_{++}$ we get that $p_h > 0$ for all h, so that for each i, either $\bar{x}_i$ satisfies the minimum-wealth condition at prices p, or $\bar{x}_i = 0$. In both cases, it follows that $p \cdot x'_i > p \cdot \bar{x}_i$ for all $x_i \in P_i(\bar{x}_i)$, $i = 1, \dots, m$. The argumentation may now be finished as in the proof of Theorem 9.2.

(4) The game form defining strategy market games is indeed an allocation process on the set $\mathfrak{E}$ of economies $\mathcal{E} = (\mathbb{R}^l_+, P_i, \omega_i)_{i=1}^m$ with $m \geq l + 1$ satisfying our standard assumptions.

We define M as the set of m-arrays $(p_i, z_i)_{i=1}^m$, where $p_i \in \triangle$ and $z_i \in \mathbb{R}^l$, and ν assigns to each $\mathcal{E} \in \mathfrak{E}$ the set of arrays $(p_i, z_i)_{i=1}^m$ which are Nash equilibria in the strategic market game, finally, h assigns to each Nash equilibrium the resulting net trades in $\mathcal{E}$.

The allocation process (ν, h) is decisive since Nash equilibria exist for the strategic market game (we may here rely on Theorem 9.2 and the existence result for Walras equilibria). It is non-wasteful since net trades which are images by h of equilibrium messages are Walras equilibrium allocations (Theorem 9.2 again) and as such Pareto optimal.

For privacy preservation, there should be correspondences ν_i depending only on characteristics (P_i, ω_i), giving equilibrium messages for economies where i has these characteristics, such that the equilibrium messages are exactly the intersection of all $\nu_i(P_i, \omega_i)$, for $i = 1, \dots, m$. If we consider economies with two three consumers and two commodities, where there are only two possible types, namely

(1) utility function $x_1 x_2^2$ and endowment $(1, 1)$,

(2) utility function $x_1^2 x_2$ and endowment $(2, 2)$

Whenever there are two of type 1 and one of type 2, there is an equilibrium with prices $(1, 1)$, but if all three are of type 1, there is only a no-trade equilibrium. This cannot be realised with correspondences ν_i of the above type.

(5) First of all, we show that the competitive allocation process in (2) and (3) gives rise to Walras equilibria: By definition, the allocation h^c satisfies $\sum_{i=1}^m h_i^c(p, z) = 0$ for all $p, z \in\ni (\mathcal{E})$, so that the allocation $(h_i^c(p, z) + \omega_i)_{i=1}^m$ is aggregate feasible. By (2), $(p, 1) \cdot h^c_(p, z) = 0$ for all i. For $i = 1, \dots, m-1$, $h_i^c(p, z) + \omega_i$ is individually optimal by (3), and since Walras law is satisfied in the sense that $\sum_{i=1}^m (p, 1) \cdot h_i^c(p, z) = 0$, we may conclude that also $h_m^c(p, z) + \omega_i$ is individually rational, so that $h^c(p, z)$ is indeed a Walras allocation.

From standard properties of Walras allocations we may now conclude that (ν^c, h^c) is non-wasteful (equilibrium allocations are Pareto optimal) and decisive (assuming $\omega_i \in \mathbb{R}^l_{++}$ for all i and strictly monotonic preferences, so that a Walras equilibria with positive prices exist). It is privacy preserving by its very definition since $\nu^c = \cap_{i=1}^m \nu_i^c(\mathcal{E})$, whereby each $\nu_i^c(\mathcal{E})$ depends only on the characteristics (P_i, ω_i) of individual i.

(6) We show first that the AGV mechanism is budget balanced, i.e. that $\sum_{i=1}^m p^A(\theta) = 0$ for all θ. This follows from

$$\sum_{i=1}^m p^A(\theta) = \sum_{i=1}^m \left[\frac{1}{m-1} \sum_{j \neq i} r_j(\theta_j) - r_i(\theta_i) \right] = \frac{m-1}{m-1} \sum_{j=1}^m r_j(\theta_j) - \sum_{i=1}^m r_i(\theta_i) = 0.$$

To show that the array $\theta = (\theta_1, \dots, \theta_m)$ of true types is a Bayesian equilibrium strategy, we choose an individual i and θ_i' arbitrarily. Then expected final payoff is

$$\mathsf{E}_{\theta_{-i}}\left[u_i(z_i^A(\theta_i',\theta_{-i}),\theta_i) - r_i(\theta_i')\right] = \mathsf{E}_{\theta_{-i}}\left[\sum_{j=1}^n u_j(z_j^A(\theta_i',\theta_{-i}),\theta_j)\right] - \mathsf{E}_{\theta_{-i}}\left[\frac{1}{m-1}\sum_{j\neq i} r_j(\theta_j)\right].$$

Here the second member on the right-hand side is independent of θ_i, and for the first member, we have that for each possible value of θ,

$$\sum_{j=1}^n u_j(z_j^A(\theta_i',\theta_{-i}),\theta_j) \le \sum_{j=1}^n u_j(z_j^A(\theta),\theta_j)$$

by the definition of z^A, so it follows that

$$\mathsf{E}_{\theta_{-i}}\left[u_i(z_i^A(\theta_i',\theta_{-i}),\theta_i) - r_i(\theta_i')\right] \le \mathsf{E}_{\theta_{-i}}\left[u_i(z_i^A(\theta) - r_i(\theta_i)\right]$$

showing that truth-telling is indeed a Bayesian equilibrium. The efficiency of the mechanism outcome follows now from the definition of z^A.