Exercises to Chapter 4

(1) An allocation in the core must be Pareto optimal. If all bundles are in the interior of $\mathbb{R}^2_+$, which they must be since they are individually rational, then marginal utilities of the individual 1 and 3 must be equal,

$$\frac{3x_{11}}{2x_{12}} = \frac{1}{2\sqrt{x_{32}}}, \tag{1}$$

and the bundle of individual 2 must satisfy

$$\frac{x_{22}}{x_{21}} = \frac{3}{2}, \tag{2}$$

and the bundles must satisfy the feasibility condition

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 6 \\ x_{12} + x_{22} + x_{32} &= 8. \end{aligned} \tag{3}$$

To this we first add the conditions of individual rationality,

$$u_1(x_{11}, x_{12}) \geq 4,\ u_2(x_{21}, x_{22}) \geq \frac{2}{3},\ u_3(x_{31}, x_{32}) = 2 + \sqrt{5}, \tag{4}$$

and for each coalition the condition that there is no improvement, that is

$$\begin{aligned} \left(4, \frac{2}{3}\right) &\notin \operatorname{int}\{(u_1(x_1'), u_2(x_2')) \mid x_1' + x_2' = (4, 3)\} \\ \left(\frac{2}{3}, 2 + \sqrt{5}\right) &\notin \operatorname{int}\{(u_2(x_2'), u_3(x_3')) \mid x_2' + x_3' = (4, 7)\} \\ \left(4, 2 + \sqrt{5}\right) &\notin \operatorname{int}\{(u_1(x_1'), u_3(x_3')) \mid x_1' + x_3' = (4, 6)\} \end{aligned} \tag{5}$$

Now, the core allocations are all $((x_{11}, x_{12}), (x_{21}, x_{22}), (x_{31}, x_{32}))$ satisfying (1)-(5).

(2) A nonempty set **N** of allocations in $\mathcal{E}$ is a von Neumann-Morgenstern solution if

(i) each $x = (x_1, \ldots, x_m)$ is feasible, i.e. $x_i \in X_i$, $i =, 1, \ldots, m$, and $\sum_{i=1}^m x_i = \sum_{i=1}^m \omega_i$.

(ii) the set **N** is internally stable w.r.t. domination, i.e. there is no $x' \in$ **N** such that for some $S \subseteq N$, $\sum_{i \in S} x_i' = \sum_{i \in S} \omega_i$ and $x_i' \in P_i(x_i)$, all $i \in S$,

(iii) the set **N** is externally stable w.r.t. domination, i.e. for each feasible allocation $x \notin$ **N**, there is an allocation $x_i'' \in N$ and a coalition $T \subseteq N$ such that $\sum_{i \in T} x_i'' = \sum_{i \in T} \omega_i$ and $x_i'' \in P_i(x_i)$, all $i \in T$.

The core of an economy $\mathcal{E}$ consists of all the non-dominated solutions, so it is internally stable, and it is also externally stable, since allocations not in the core must be dominated. It follows that the economy has a nonempty von Neumann-Morgenstern solution if it has a nonempty core, cf. Thm.4.2.

(3) For each k, an allocation x^k in the k-replica economy $\mathcal{E}^k$ induces "equal-treatment" allocations x^s in all the economies $\mathcal{E}^{ks}$ for $s \in \mathbb{N}$, so that in this sense x^k is an allocation in $\mathcal{E}^{ks}$ for all s. We show that if an allocation x^k in $\mathcal{E}^k$ is in the core of $\mathcal{E}^{ks}$ for all $s \in \mathbb{N}$, then it is (genuinely) equal-treatment. We shall need that preferences P_i for $i = 1, \dots, m$ are complete so that for each i and each pair (x_i, x_i') of bundles in X_i, either $x_i' \in \operatorname{cl} P_i(x_i)$ or $x_i \in \operatorname{cl} P_i(x_i')$.

Suppose that $x \in \operatorname{Core}(\mathcal{E}^k)$ but is not equal-treatment, so that there are at least two agents i_1, i_2 with identical characteristics such that $x_{i_1}^k \neq x_{i_2}^k$.

Assume that $x_{i_2}^k \in \operatorname{cl} P_i(x_{i_1}^k)$. By strict convexity, we then have that

$$x^* = \frac{1}{2} x_{i_2}^k + \frac{1}{2} x_{i_1}^k \in P_i(x_{i_1}^k).$$

In $\mathcal{E}^{2k}$, the allocation where each of the bundles x_i^k for $i = 1, \dots, m$ occurs twice can be improved via the coalition consisting of the two copies of i_1 together with two one of each individual representing the bundles x_i^k for $i \neq i_1, i_2$: If the two copies of i_1 get x^i and the other participants keep their bundles, this allocation is feasible for the coalition, and upon some redistributing it is possible to assign preferred bundles to all members of the coalition. It follows that the "equal-treatment" version of x^k does not belong to $\operatorname{Core}(\mathcal{E}^{2k})$.

(4) A Walras equilibrium in the coalition production economy $\mathcal{E} = ((X_i, P_i, \omega_i)_{i=1}^m, \mathcal{Y})$ is an array (x, y, p, π), where $x = (x_1, \dots, x_m) \in \prod_{i=1}^m X_i$ is an array of indvidually feasible bundles, $y \in \mathcal{Y}(N)$ a production vector feasible for the grand coalition, $p \in \triangle$ a price system, and $\pi = (\pi_1, \dots, \pi_m) \in \mathbb{R}_+^m$ an array of profit shares, which satisfy

(i) aggregate feasibility, i.e. $x \sum_{i=1}^m x_i = \sum_{i=1}^m \omega_i + y$,

(ii) Individually optimal given p and π: for each i, $p \cdot x_i \leq p \cdot \omega_i + \pi_i$, and if $x_i' \in P_i(x_i)$, then $p \cdot x_i' > p \cdot \omega_i + \pi_i$, and $p \cdot y = \max_{y' \in \mathcal{Y}(N)} p \cdot y'\}$,

(iii) no coalitional improvement: there is no $S \subseteq N$ and $y^S \in \mathcal{Y}(S)$, $(\pi_i^S)_{i \in S} \in \mathbb{R}_+^S$ such that $\sum_{i \in S} \pi_i^S = p \cdot y^S$ and $p \cdot x_i'' \leq p \cdot \omega_i + \pi_i^S$ for some $x_i'' \in P_i(x_i)$, all $i \in S$.

For the second part, we first notice that if (x, y, p, π) is a Walras equilibrium, then π belongs to the core of the TU game (N, v) with

$$v(S) = \max_{y \in \mathcal{Y}(S)} p \cdot y.$$

Indeed, if there is S with $v(S) > \sum_{i \in S} \pi_i$, then there is $y^S \in \mathcal{Y}(S)$ and $(\pi_i^S)_{i \in S}$ with $\sum_{i \in S} \pi_i^S = p \cdot y^S$ and $\pi_i^S > \pi_i$ for each $i \in S$, and we get a contradiction of (iii).

If (x, y, p, π) is a Walras equilibrium and x does not belong to the core of $\mathcal{E}$, then there must be $S \subseteq N$ and $(x_i')_{i \in S}$ such that $\sum_{i=1}^m (x_i - \omega_i) \in \mathcal{Y}(S)$ and $x_i' \in P_i(x_i)$ for all $i \in S$. By the properties of a Walras equilibrium, we have that

$$p \cdot (x_i' - \omega_i) > \pi_i, \text{ all } i \in S,$$

and it follows that

$$v(S) \geq p \cdot \sum_{i \in S} (x_i' - \omega_i) > \sum_{i \in S} \pi_i$$

contradicting that π belongs to the core of (N, v).

(5) Consider an economy with 2 goods and 3 consumers, all having consumption set $\mathbb{R}^2_+$ and endowment $(2, 2)$. The preferences can be described by utility functions which are positively homogeneous (of first degree) and the set of bundles having utility 1 is defined as

(i) Consumer 1: The line segment from $(2, 2)$ to $(1, 3)$ together with the vertical half-line from $(1, 3)$ and the horizontal half-line from $(2, 2)$,

(ii) Consumer 2: The line segment from $(2, 2)$ to $(3, 1)$ together with the vertical half-line from $(2, 2)$ and the horizontal half-line from $(3, 1)$,

(iii) Consumer 3: The set $\{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 x_2 = 1\}$.

Then the allocation $((1, 3), (3, 1), (2, 2))$ defines a Walras equilibrium together with the price vector $(1, 1)$, consequently the allocation belongs to the core. All the marginal contributions of the consumers are 4, so the allocation is a value allocation as well.

(6) The economy $\mathbb{R}^3_+, P, \omega_i)_{i=1}^3$ considered in Box 4 satisfies the Assumptions 0.1 and 0.2. As shown in Box 4, the allocation $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \in \mathcal{F}(\mathcal{E})$ belongs to the bargaining set of $\mathcal{E}$ but is not in the core, indeed the coalition $\{1, 2\}$ has an improvement of x. But then the coalition consisting of k copies of 1 and of 2 has an improvement of the equal-treatment allocation x^k in $\mathcal{E}^k$ induced by k, consequently x^k is not in the core of $\mathcal{E}^k$.