Solutions to Exercises in

Game Theory
Chapter 10

1. Since $u_1(\sigma) + u_2(\sigma) = 0$ for every array of (mixed or pure) strategies, any payoff is Pareto optimal in the sense that the grand coalition cannot increase the payoff of one player without reducing that of the other player.

Under standard assumptions, the game Γ has Nash equilibrium in mixed strategies. Since the coalition $\{1, 2\}$ cannot improve any strategy array, any Nash equilibrium is automatically strong Nash equilibrium.

The argument does not extend to zero-sum games with more than two players: The grand coalition cannot improve, but there are now several two-player coalitions which may have a joint strategy which gives its members a larger payoff than the Nash equilibrium.

2. Consider the tree-person game given below,

I	L	R
U	$(1, 1, 0)$	$(10, 0, 0)$
D	$(0, 10, 0)$	$(9, 9, 0)$

II	L	R
U	$(1, 1, 2)$	$(10, 0, 1)$
D	$(0, 10, 1)$	$(9, 9, 1)$

where player 1 chooses row, player 2 column and player 3 matrix.

There is a single Nash equilibrium, namely (U, L, II), where each player uses a strictly dominating strategy. There is no strategy combination which (weakly) Pareto dominates the Nash equilibrium payoff $(1, 1, 2)$, so improvements can be obtained only through the coalitions with two players, and indeed only through the coalition $\{1, 2\}$, which may choose (D, R) given that player 3 chooses II. This improvement is however not internally consistent, since each of the players may defect gainfully given this strategy choice.

3. The strategy array (T, L, M_1) is a Nash equilibrium, since none of the players can improve by defecting. It is Pareto optimal since there is no payoff array which give the each of the players at least as in $(1, 1, -5)$ and at least one of them more.

The same reasoning shows that the strategy array (B, R, M_1) is a Pareto optimal Nash equilibrium. Both of these are strong Nash equilibria since none of the two-player coalitions can obtain something better given the choice of the third player.

4. (N, V_s) is a cooperative NTU game: Closedness of $V_s(S)$ follows directly from its definition, as does comprehensiveness (if $(z_i)_{i\in S} \in V_s(S)$ and $z_i' \leq z_i$ for all $i \in S$, then clearly

$(z'_i)_{i\in S} \in V_s(S)$). Finally, $V_s(S)$ is upper bounded if the the set of payoff arrays

$$\{(u_i(t_S, s_{N\setminus S}))_{i\in S} \mid t_S \in \Sigma^S\}$$

is bounded (which is the case if each S_i is a finite set).

Suppose that s is not a strong Nash equilibrium of Γ. Then there is a coalition S and an S-strategy array $t_S \in \Sigma^S$ such that $u_i(t_S, s_{N\setminus S}) > u_i(s)$ for $i \in S$. But $V_s(S)$ contains the payoff vector $(u_i(t_S, s_{N\setminus S}))_{i\in S}$, so that $u(s)$ does not belong to the core of (N, V_s). Conversely, if $u(s)$ is not in the core of (N, V_s), then there is a coalition S and a payoff vector $(u'_i)_{i\in S}$ from $V_s(S)$, such that $u'_i > u_i(s)$ for all $i \in S$. Using the definition of V_s, we have that there is an S-strategy t_S such that $u_i(t_S, s_{N\setminus S}) > u_i(s)$ for all $i \in S$, meaning that s cannot be a strong Nash equilibrium.

Let s be a strong Nash equilibrium such that $u(s)$ is Pareto optimal in $V(N)$. If $u(s)$ is not in the core of (N, V_β), then there is a coalition $S \neq N$ and a payoff vector $(u'_i)_{i\in S} \in V_\beta(S)$ such that $u'_i > u_i(s)$ for all $i \in S$. Using the definition of V_β, we get that for every $N\setminus S$-strategy $s'_{N\setminus S}$, in particular for the $N\setminus S$-strategy $s_{N\setminus S}$, there is an S-strategy t'_S such that $u_i(t'_S, s_{N\setminus S}) > u_i(s)$, all $i \in S$, contradicting that s is a strong Nash equilibrium.

5. If $k = 1$, there is only one set C, equal to N, and the Nakamura-number is $+\infty$. Assume $k < n$. It is clearly enough to consider sets C of cardinality k. To find the smallest family of such sets with empty intersection, we may instead look for the smallest number of their complements, that is $(n-k)$-sets, covering N. This number is $\left\lceil \frac{n-k}{n} \right\rceil$, which is therefore the Nakamura number of $\mathcal{W}$.

6. The unanimity games $\mathcal{W}_i$ with $C' \in \mathcal{W}_i$ if and only if $i \in C'$ are weighted majority games (where i has weight $> \frac{1}{2}$) for each $i \in C$. If $\mathcal{W}_1$ and $\mathcal{W}_\in$ are unanimity games with minimal winning coalition C_1 and C_2 respectively, $C_1 \cap C_2 = \emptyset$, then the inner compound of $\mathcal{W}_1$ and $\mathcal{W}_2$ using the weighted majority game on $\{1, 2\}$ with weights $\left(\frac{1}{2}, \frac{1}{2}\right)$ is a unanimity game with minimal winning coalition $C_1 \cup C_2$.

If $|C| = s$, then repeated splitting of coalitions into two disjoint sets of cardinality differing by at most one will result in singletons after at most m steps, where m is the smallest number such that such that

$$\frac{1}{2^m} \geq s.$$

from which we get that $m = \lceil \log_2 s \rceil$.