Solutions to Exercises in

Game Theory
Chapter 11

1. A bargaining problem consists of a subset B of $\mathbb{R}^2$ and a point $a \in \mathbb{R}^2$ satisfying (i)-(iv) of Defn.1. For the disagreement payoff vector a, since no other information is available it is natural to choose the vector of minmax payoffs,

$$a = (4, 2)$$

(the smallest payoff to player 1 is obtained if player 2 uses the mixed strategy with equal weights 1/2 on both columns, and player 2 can obtain at most 2 if player 1 chooses T). The set B then takes the form

$$B = \left[\mathrm{conv}(\{(6,3),(2,4),(3,2),(7,0)\}) - \mathbb{R}^2_+\right] \cap \left[\{(4,2)\} + \mathbb{R}^2_+\right].$$

To find the Nash bargaining solution we maximize $(u_1 - 4)(u_2 - 2)$ on B. The maximum is attained at $(6, 3)$.

2. We assume as usual that the disagreement point is $(0, 0)$. The egalitarian solution Φ_E is Pareto optimal if the boundary of B has no segments which are parallel to the coordinate axes. It satisfies symmetri since $\Phi_1(B) = \Phi_2(B) = \cdots = \Phi_n(B)$ for all bargaining problems B. Finally, strong monotonicity follows since $\Phi_E(B)$ is the upper bound of the intersection of B and the diagonal.

For arbitrary $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{++}$, the bargaining solution Φ finds the element of B where the boundary intersects the ray defined by $(\lambda_1^{-1}, \dots, \lambda_n^{-1})$. The solution satisfies Axiom 1 (Pareto) and 2 (independence of affine transformation), it clearly violates symmetry which $(\lambda_1, \dots, \lambda_n) \neq (1, \dots, 1)$. It satisfies Axiom 4 as the intersection of the ray and the smaller bargaining set containing the solution of the larger set. The monotonicity axiom 5 is satisfied by Φ, and so is Axiom 6, since it reduces to considerations of points on the fixed ray defined by $(\lambda_1^{-1}, \dots, \lambda_n^{-1})$.

3. For the given B we must show that there is $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2_{++}$ such that the ray given by λ is a normal to B at its intersection with the boundary of B. This follows since otherwise for each $\lambda' \in \mathbb{R}^2_{++}$ the tangent at the intersection point must intersect B either to the right or to the left of the normal. Since the intersection cannot be to the right side at one endpoint and to the left on the other, continuity of the intersection correspondence and connectedness of the boundary gives that there must be a point where this intersection is empty. (This point may be one of the endpoints – indeed both endpoints satisfy the condition, and in this case either

$\lambda_1 = 0$ or $\lambda_2 = 0$. Examples show that there are bargaining problems B where the condition can be satisfied only at the endpoints.)

4. Pareto optimality is satisfied: Suppose that $\Phi_R(B)$ is not Pareto optimal, then there must be some $a \in B$ with $a_i \geq \Phi_R(B)_i + \varepsilon$ for at least one i and some $\varepsilon > 0$, and in then $a_i(\{x \in B \mid x_j \geq \Phi_R(B)_j, \text{ all } j\}) > \Phi_R(B)_i$. By continuity we get that $z_i^j < a_i - \varepsilon$ for all j, a contradiction.

Independence of affine transformation is a consequence of the definition. For symmetric problems B, we have that $z_1^j = \cdots = z_n^j$ for all j, so that $\Phi_R(B)_1 = \cdots = \Phi_R(B)_n$. We show by an example that independence of irrelevant alternatives is *not* satisfied: Let $B = \{x \in \mathbb{R}^2_+ \mid x_1 + x_2 = 1\}$ with $\Phi_R(B) = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$. Then $\Phi_R(B)$ belongs to the set

$$B' = \left\{x \in \mathbb{R}^2_+ \,\middle|\, x_1 + x_2 = 1, x_1 \leq \frac{1}{2}\right\}.$$

For B', we find $z^1 = \left(\frac{1}{4}, \frac{1}{2}\right)$, $z^2 = z^3 = \cdots = \left(\frac{3}{8}, \frac{5}{8}\right)$, so that $\Phi_R(B') \neq \Phi_R(B)$.

Next, we check the monotonicity axiom for 2-person bargaining: If $a_1(B') = a_1(B)$ and $g_B \leq g_{B'}$, then $z_2^1(B') \geq z_2^1(B)$, and similarly $z_2^j(B') \geq z_2^j(B)$ for all j, so that $\Phi_R(B)_2 \geq \Phi_R(B)_2$, so that monotonicity holds. Finally, the construction of the points $z^1, z^2, \ldots$ for the sum $B + B'$ of two bargaining problems B and B^2 satisfies

$$z_i^j(B + B') \geq z_i^j(B) + z_i^j(B')$$

for each j, so that $\Phi_R(B + B') \geq Phi_R(B) + \Phi_R(B')$ and Φ_R satisfies superadditivity.

5. In the first step we consider bargaining problems of the form $B = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n p_i x_i \geq r\}$ for some $p = (p_1, \ldots, p_n) \in \mathbb{R}^n_{++}$. We claim that the Nash bargaining solution is the midpoint of the simplex $\{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n p_i x_i = r\}$. Indeed, first order conditions for maximum of $x_1 x_2 \cdots x_n$ under the constraint $\sum_{i=1}^n p_i x_i = r$ gives

$$(x_1, \ldots, x_n) = \sum_{i=1}^n \frac{1}{n}\left(0, \ldots, 0, \frac{r}{p_i}, 0, \ldots, 0\right),$$

which is indeed the midpoint of $\{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n p_i x_i \geq r\}$.

Next, let B be arbitrary and let $a = \Phi(B)$ be a point on the boundary of B such that a is a midpoint of the intersection of a tangent plane to bd$A$ and the positive orthant (such a point exists by the first step, since it is found by maximizing $x_1 \cdots x_n$ on A). By midpoint domination, a is the solution to the bargaining problem defined by the tangent plane, and by independence, it is also the solution to the problem B. Uniqueness is obvious since there cannot be two distinct points a with the above property.

6. The Equal Area bargaining solution satisfies the Pareto axiom by construction (if bd$B$ contains no vertical or horizontal segments), and it satisfies scale invariance. If a bargaining problem is symmetric, then the diagonal splits it into two sets with equal area, so also Axiom 3 is satisfied. Axiom 4 is violated, since the equal area property may be violated when considering subsets of B. Monotonicity may be violated as well, since the increase from g_B to $g_{B'}$ man take place to the right of the line through $\Phi(B)$, so that $\Phi(B')_2 < \Phi(B)_2$. Superadditivity is satisfied since the division into equal areas is preserved by addition.