Solutions to Exercises in

Game Theory
Chapter 13

1. For the coalition structure $\{\{1\}, \{2\}, \{3\}\}$, the only feasible payoff vector $(0, 0, 0)$ (together with the coalition structure) belongs to the bargaining set, since there are no objections (no two individuals in the same coalition of the structure).

For $\{\{1, 2\}, \{3\}\}$, the vector $(5, 15, 0)$ is in the bargaining set: Objections of 1 against 2 must involve $\{1, 3\}$ giving 3 less than 25, so that so that 2 has a counterobjection, aind conversely objections of 1 against 2 have counterobjections. For all other payoff vectors there are objections without counterobjections (by the same argument).

For $\{\{1, 3\}, \{2\}\}$ the same reasoning singles out $(5, 0, 25)$ as belonging to the bargaining set, and for $\{\{2, 3\}, \{1\}$ we obtain the payoff vector $(0, 15, 25)$. For the coalition structure $\{\{N\}\}$, payoff vectors (x_1, x_2, x_3) with $x_1 + x_2 + x_3 = 41$ and $x_i + x_j < v(\{i, j\})$ (such as $(4, 14, 24)$) are in the bargaining set, since each objection as a counterobjection.

2. The upper vectors b^v_i of the bankruptcy game v are given by

$$b^v_i = E - \max\left\{E - \sum_{j\notin N\setminus\{i\}} c_j, 0\right\} = \begin{cases} c_i & c_i \le E \\ E & \text{otherwise,} \end{cases}$$

for $i \in N$. For any coalition S, the gap function is then given by

$$g^v(S) = \sum_{i\in S} b^v_i - v(S) = \sum_{i\in S} b^v_i - \max\left\{E - \sum_{j\notin S} c_j, 0\right\},$$

and $\lambda^v_i = \min_{S:i\in S} g^v(S) = g^v(N)$.

Assume $c_i < E$ for all i. Now the τ-value is found as the point on the line segment between $b^v - \lambda^v = \left(\max\left\{\sum_{j\neq i} c_j - E, 0\right\}\right)_{i\in N}$ and c at which the sum of coordinates is E.

If $\sum_{j\neq i} c_i > E$ for all i, then $m_i(N, E, c) = b^v_i - \lambda^v_i = 0$, $\hat{m}_i = c_i$, since $c_i < E$, and we have that $b^v - \lambda^v = 0$, and $A(N, E, c) = P(N, E, c)$. If $\sum_{j\neq i} c_j < E$ for some i, then $b^v_i - \lambda^v_i = m_i(N, E, c) \neq 0$ and the τ-value is found on the segment between $m(N, E, c)$ and c, which again is $A(NE, c)$.

If $c_i > E$ for some i, the analysis proceeds similarly.

3. First of all, we check the Talmud rule in the simple case where $N = \{1, 2\}$. We claim that with the game defined by

$$v(\{i\}) = \max\{E - c_j, 0\}, i, j = 1, 2, i \neq j, v(N) = E,$$

the Talmud rule can be written as

$$\phi_i = v(N) - v(\{j\}) + \frac{v(N) - v(\{1\} - v(\{2\})}{2}. \tag{1}$$

There are four possible cases: If $\frac{c_1}{2} + \frac{c_2}{2} \geq E$, then either (i) $c_1 > E, c_2 > E$, so that $v(\{i\}) = 0$, $i = 1, 2$, (**??**) becomes equal to $E/2$, which is $T_i(\{1,2\}, E, c) = \lambda$ with $\lambda = E/2 < c_i$. If (ii) $c_1 > E, c_2 < E$ (the case $c_1 > E, c_2 > E$ is treated similarly), then the expression (**??**) becomes

$$\phi_1 = E + \frac{E - c_2 - E}{2} = E - \frac{c_2}{2}, \phi_2 = E - (E - c_2) + \frac{E - c_2 - E}{2} = \frac{c_2}{2},$$

which again is $T_i(\{1,2\}, E, c)$ with $\lambda = E - \frac{c_2}{2} < \frac{c_1}{2}$. If $\frac{c_1}{2} + \frac{c_2}{2} \leq E$, then the case (iii) $c_1 > E, c_2 < E$ we get the same expression for $T_i(\{1,2\}, E, c)$, but now with $\lambda = c_1 - \left(E - \frac{c_2}{2}\right)$, and finally (iv) $c_1 < E, c_2 > E$, in which case we get

$$\phi_1 = E - (E - c_1) + \frac{E - c_1 + E - c_2 - E}{2} = c_1 - \frac{E - (c_1 + c_2)}{2}, \phi_2 = c_2 - \frac{E - (c_1 + c_2)}{2},$$

which is $T_i(\{1,2\}, E, c)$ with $\lambda = \frac{E-(c_1+c_2)}{2}$.

Next, we show that the Talmud rule satisfies consistency considered as solution to the bankruptcy game.

Suppose first that $\sum_{i \in N} c_i \geq E$, choose an arbitrary player i^0. Then the reduced bankruptcy problem is one where the estate has been reduced to

$$E^{i^0} = E - \min\left\{\frac{c_{i^0}}{2}, \lambda\right\},$$

where λ is the balancing factor securing that $\sum_{i \in N} \min\left\{\frac{c_i}{2}, \lambda\right\} = E$. By the definition of λ we have that $\sum_{i \neq i^0} c_i = E^{i^0}$. Then the Talmud rule applied to $(N\backslash\{i^0\}, E^{i^0}, (c_i)_{i \neq i^0})$ gives the result $\min\left\{\frac{c_i}{2}, \lambda^{i^0}\right\}$, where λ^{i^0} is such that the sum equals E^{i^0}. Clearly, $\lambda^{i^0} = \lambda$, and it follows that $T_i(N\backslash\{i^0\}, E^{i^0}, (c_i)_{i \neq i^0}) = T_i(N, E, c)$ for all $i \neq i^0$. The case of $\sum_{i \in N} c_i \leq E$ is treated in the same way.

We now have that the Talmud rule coincides with the prenucleolus for $|N| = 2$ and satisfies consistency, so by consistency it coincides with the prenucleolus.

4. For the payoff vector $z^0 = \left(\frac{1}{7}, \ldots, \frac{1}{7}\right)$, the excess of a coalition S containing one of the specific mentioned sets is $1 - \dfrac{|S|}{7}$, and for any other coalition it is $-\dfrac{|S|}{7}$, consequently $s_{ij}(z^0) = \frac{4}{7}$ for each pair (i, j) with $i \neq j$, and z^0 belongs to the kernel.

Next consider the payoff vector $z^1 = \left(\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}, 0, \ldots, 0\right)$. Here the excesses are 0 for coalitions containing $\{1, 2, 4\}$, $\frac{1}{3}$ for coalitions containing one of the designated sets but only two members of $\{1, 2, 4\}$, and $\frac{2}{3}$ for coalitions containing a designated set but only one element of $\{1, 2, 4\}$. For each pair (i, j) there is a designated set with excess $\frac{2}{3}$ containing i but not j, so $s_{ij}(z^1) = \frac{1}{3}$ for all pairs (i, j), and consequently z^1 belongs to the kernel.

For any $\lambda \in [0, 1]$, the payoff vector $z^\lambda = \lambda z^0 + (1 - \lambda)z^1$ assigns equal payment in the interval $\left[\frac{1}{7}, \frac{1}{3}\right]$ to players 1,2 and 4, and equal payment in $\left[0, \frac{1}{7}\right]$ for the other players. The largest excesses still obtain for coalitions containing only one element of $\{1, 2, 4\}$, and all $s_{ij}(z^\lambda)$ have the same size, so that also z^λ belongs to the core.

It is easily seen that the same reasoning applies to any of the seven designated sets, showing that for $j = 2, \ldots, 7$, the payoff vector z^j and the line segment $[z^0, z^j]$ belong to the kernel.

For all other payoff vectors, there will be some player outweighing some other player, so that only the payoff vectors exhibited above can be in the kernel of v.

5. We first check that $z = (3, 2, 1)$ belongs to the kernel of v, we find the excesses

$$e(\{1\}, z) = -3, e(\{2\}, z) = -2, e(\{3\}, z) = -1,$$
$$e(\{1, 2\}, z) = -1, e(\{1, 3\}) = -1, e(\{2, 3\}, z) = -1, e(N, z) = 0,$$

and we find that $s_{ij}(z) = -1$ for all pairs (i, j) with $i \neq j$. Clearly, there is no case where i outweighs j, so that $(3, 2, 1)$ belongs to the kernel.

To see that $(3, 2, 1)$ is the only element of the kernel, consider an imputation where some player i gets more than in z. Then the excess of some coalition containing i must decrease whereas the excess of some coalition not containing j must increase, and consequently j outweights i, so that the imputation cannot belong to the kernel.

The reduced games are are:

$$(\{2, 3\}, v_z^1) : v_z^1(\{2\}) = 1, v_z^1(\{3\}) = 0, v(\{2, 3\}) = 3, e(\{2\}, (2, 1)) = -1, e(\{3\}, (2, 1)) = -1,$$
$$(\{1, 3\}, v_z^2) : v_z^2(\{1\}) = 2, v_z^2(\{3\}) = 0, v(\{1, 3\}) = 4, e(\{1\}, (2, 1)) = -1, e(\{3\}, (2, 1)) = -1,$$
$$(\{1, 2\}, v_z^3) : v_z^3(\{1\}) = 2, v_z^3(\{2\}) = 1, v(\{1, 2\}) = 5, e(\{1\}, (2, 1)) = -1, e(\{2\}, (2, 1)) = -1,$$

and with the same arguments as above, it is seen that $(3, 2, 1)$ induces the unique kernel element in the three reduced games.

6. For the graph game, the upper vector b^v is given as $b_i^v = \sum_{h,k\in N} w_{hk} - \sum_{h,k\in N\setminus\{i\}} w_{hk} = \sum_{k\in N} w_{ik}$ for $i \in N$. The gap function $g^v(S)$ takes the form

$$g^v(S) = \sum_{j\in S} b_j^v - v(S) = \sum_{h\in S}\sum_{k\in N} w_{hk} - \sum_{h,k\in S} w_{hk} = \sum_{h,k} \in S + \sum_{h\in S, k\notin S} w_{hk}, \tag{2}$$

in particular, $g^v(N) = \sum_{h,k\in N} w_{hk} > 0$. The vector λ^v is found as

$$\lambda_i^v = \min_{S:i\in S} g^v(S) = g^v(\{i\}) = b_i^v,$$

since the minimum in (**??**) is attained where the coalition is as small as possible, and we find the τ-value as

$$\tau_i(v) = b_i^v - \frac{g^v(N)}{\sum_{i\in N} b_i^v} b_i^v = \sum_{j\in N} w_{ij} - \frac{\sum_{h,k\in N} w_{hk}}{2\sum_{h,k\in N} w_{hk}} \sum_{j\in N} w_{ij} = \frac{1}{2}\sum_{j\in N} w_{ij}.$$