Solutions to Exercises in

Game Theory
Chapter 14

1. Assume that (N, V) is convex, and let $S = \{i_1, \dots, i_k\}$ be an arbitrary coalition. We must find an element of Core(N, V) which belongs to the boundary of S.

By Lemma 1 in Section 14.2 which tells us that $(N\backslash\{i\}, V^{)i(})$ is convex for each $i \in N$. In the proof of Theorem 1, we have that if x^1 belongs to the core of $(N\backslash\{i_1\}, V^{)i_1(})$, then (x^1, a_{i_1}) with $a_{i_1} = \sup V(\{i_1\})$ is in the core of (N, V), and clearly a_{i_1} belongs to the boundary of $V(\{i_1\})$. Proceeding with i_2, we find x^2 such that (x^2, a_{i_2}) is in the core of $V^{(i_1}$, and consequently (x^2, a_{i_2}, a_{i_1}) is in the core of (N, V), and $a_2 = \sup V^{)i_1(}(\{i_2\})$. It follows that (a_{i_2}, a_{i_1}) is on the boundary of $V(\{i_1, i_2\})$. Proceeding in this way, we find a core element $(x_{N\backslash S}, x_S)$ for (N, V) such that x_S belongs to the boundary of $V(S)$.

2. (a) The game $(\{1, 2, 3\}, V)$ with $V(\{2\} = \{x \in \mathbb{R} \mid x \le 1\}$, $V(\{1, 2\} = \{(x_1, x_2) \in \mathbb{R}2 \mid x_2 \le 1\}$, $V(\{2, 3\}) = \{(x_2, x_3) \in \mathbb{R}2 \mid x_2 \le 1\}$, $V(\{1, 2, 3\} = \{x \in \mathbb{R}^3 \mid x_i \le 1, i = 1, 2, 3\}$, and $V(S) = \mathbb{R}^S_-$ otherwise, is convex with

$$\widetilde{V}(\{1, 2\}) \cap \widetilde{V}(\{2, 3\}) \subset \widetilde{V}(\{2\}),$$

but the balanced family $\{\{1, 2\}, \{3\}\}$ and the payoff vector $x = (3, 1, 0)$ satisfies $x \in \widetilde{V}(\{1, 2\}) \cap \widetilde{V}(\{3\})$ but $x \notin V(\{1, 2, 3\})$.

(b) The game $(\{1, 2, 3\}, V)$ with $V(\{i\}) = \{x \mid x \le 1\}$ for $i = 1, 2, 3$, $V(S) = \mathbb{R}^S_-$ for $|S| = 2$ and $V(\{1, 2, 3\}) = \{x \in \mathbb{R}^3 \mid \sum x_i \le 3\}$ fails to be convex, since e.g. $\widetilde{V}(\{1\} \cup \widetilde{V}(\{2\}) = \{x \mid x_1 \le 1, x_2 \le 1)$ fails to be contained in $V(\{1, 2\}) = \mathbb{R}^2_-$. It is balanced since the only nontrivial balanced family is $\{\{1\}, \{2\}, \{3\}\}$ and $\cup_i \widetilde{V}(\{i\}) = \{x \in \mathbb{R}^3 \mid x_i \le 1, i = 1, 2, 3\} \subset V(\{1, 2, 3\})$.

(c) Consider the game $(\{1, 2, 3\}, V)$ with $V(S) = \{x \in \mathbb{R}^S \mid \sum_{i\in S} x_i^2 \le 1\}$ for $S \ne \{1, 2, 3\}$ and $V(\{1, 2, 3\}) = \{x \in \mathbb{R}^3 \mid x_i \le 1, i = 1, 2, 3\}$. For each π, one can find a family $\mathcal{B}$ of coalitions which is π-balanced and contains at most two singleton coalitions, so that $\cap_{S\in\mathcal{B}} \tilde{V}(S)$ contains a point x such that $x_i > 1$ for some i with $\{i\} \notin \mathcal{B}$. Since $x \notin V(\{1, 2, 3\})$, the game is not π-balanced. However, $(1, 1, 1)$ belongs to the core of $(\{1, 2, 3\}, V)$, which therefore is nonempty.

3. If $x \in$ Core(N, v_q) then $\sum_{i\in S} x_i \ge v_q(S) = \sup\{q_S \cdot x'' \mid x'' \in V(S)\}$. If $x \notin$ Core(N, V), then there is $S \subset N$ and $x' \in V(S)$ such that $x'_i > x_i$ for all $i \in S$, so that $\sum_{i\in S} q_{S,i} x'_i > q_S \cdot x_S > \sup\{q_S \cdot x'' \mid x'' \in V(S)\}$ contradicting that $x' \in V(S)$. We conclude that $x \in$ Core(N, V).

To obtain the converse statement we need that for each coalition $S \subset N$, the projection of the Core(N, V) on $\mathbb{R}^S$ can be separated from $V(S)$ by the linear form q^S.

4. The extended core is nonempty if the game (N, V_λ) where $V(N)$ is blown up by the factor $\lambda > 1$ to $\lambda V(N)$ has a nonempty core. Therefore conditions for a nonempty extended core can be obtained from standard conditions on (N, V_λ): Suppose that there are π and λ such that for every π-balanced family $\mathcal{B}$ of coalitions,

$$\cap_{S\in\mathcal{B}}\widetilde{V}(S) \subseteq \lambda V(N).$$

Then $\text{Core}_e(N, V) \neq \emptyset$.

5. Let $(x_1, \dots, x_n)$ be an allocation such that the corresponding equal-treatment allocation is in the core of any replica economy, and consider the set

$$P = \{x_i' - \omega_i \mid u_i(x_i') > u_i(x_i), i \in N\}.$$

Suppose that $0 \in \text{conv}\, P$. Then is $S \subset N$ and $\lambda_i > 0$ for $i \in S$ with $\sum_{i\in S} \lambda_i = 1$ such that

$$\sum_{i\in S} \lambda_i(x_i' - \omega_u) = 0.$$

By continuity of the utility functions u_i, we may choose x_i'' close to x_i' such that $u_i(x_i'') > u_i(x_i)$ for $i \in S$ and all the weights in the convex combination are rational numbers with common denominator N, i.e. such that

$$\sum_{i\in S} \frac{s_i}{N}(x_i' - \omega_u) = 0$$

with $s_i \in \mathbb{N}$, $s_i \in S$, and $N \in \mathbb{N}$. Choose now the replica economy with N agents of each type, and let S_N be a coalition in this economy consisting of s_i copies of the ith type. Then S_N has an improvement of the equal-treatment allocation defined by $(x_1, \dots, x_n)$, a contradiction, and we conclude that $0 \notin \text{conv}\, P$.

Using monotonicity of u, we have that $\text{conv}\, P \cap \mathbb{R}_- = \emptyset$, and by separation of convex sets there is $p \in \mathbb{R}^l_+$, $p \neq 0$, such that

$$p \cdot (x_i' - \omega_i) > 0 \text{if } u_i(x_i') > u_i(x_i)$$

for $i = 1, \dots, n$. It is easily checked that $p \cdot (x_i - \omega_i) = 0$ for each i, so that $(x_1, \dots, x_n, p)$ is a equilibrium.

6. [Warning: There is a typo in the definition of $V(S)$, which should be

$$V(S) = \left\{(z_i)_{i\in S} \,\middle|\, \exists\, x_i \in X_i, y_i \in Y_i, z_i \leq u_i(x_i), i \in S \,:\, \textstyle\sum_{i\in S}(x_i - y_i) = \sum_{i\in S} \omega_i\right\}$$

(the last sum is over members of S only)]

Let (N, V) be the market game with market $\mathcal{E} = (X_i, Y_i, \omega_i, u_i)_{i\in N}$. Let C be a balanced family of coalitions with balancing weights $(\lambda_S)_{S\in C}$. If $z_S \in V(S)$ for each S, then there are x_i^S, y_i^S for $i \in S$ such that $\sum_{i\in S}(x_i^S - y_i^S) = \sum_{i\in S} \omega_i$. For each $i \in N$, let $z_i^N = \sum_{S\in C} \lambda_S z_i^S$, $x_i^N = \sum_{S\in C} \lambda_S x_i^S$ and $y_i^N = \sum_{S\in C} \lambda_S y_i^S$. Then $x_i^N \in X_i, y_i^N \in Y_i$ by convexity of X_i and $u_i(x_i^N) \geq z_i^N$ by convexity of the utility functions u_i, each $i \in S$. Moreover

$$\begin{aligned}\sum_{i\in N}(x^N - y^N) &= \sum_{i\in N}\sum_{S\in C} \lambda_S(x_i^S - u_i^S) \\ &= \sum_{S\in C}\sum_{i\in S} \lambda_S(x_i^S - y_i^S) = \sum_{S\in C}\sum_{i\in S} \lambda_S\omega_i = \sum_{i\in N} \omega_i\end{aligned}$$

from which we get that $\sum_{S\in C} \lambda\widetilde{V}(S) \subset V(N)$, so that $V(N)$ is cardinally balanced.

The second part follows directly from the first one, since the restriction of V to subcoalitions of S is the market game associated with the market $(X_i, Y_i, \omega_i, u_i)_{i\in S}$.